NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2014–2015
MH1810 – Mathematics 1

NOVEMBER 2014 TIME ALLOWED: 2 HOURS

Matriculation Number: | | | | | | | | | |

Seat Number: | | | | | |

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **SIXTEEN (16)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the attachments.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Questions	Marks
1 (8)	
2 (7)	
3 (7)	
4 (8)	

Questions	Marks
5 (15)	
6 (15)	
7 (20)	
8 (20)	

Total (100)	

QUESTION 1. **(8 Marks)**

(a) Express the complex number $z = 1 - i$ in polar form $z = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $\theta \in (-\pi, \pi]$.

(b) Using part (a), express the complex number $(1-i)^7$ in the form $x + yi$, where x and y are positive integers.

QUESTION 2. **(7 Marks)**

(a) Given that the line ℓ passing through the origin O and $A\,(1, 1, \alpha)$ is parallel to the plane $\Pi : x + y + z = 1$, find the value of the constant α.

(b) Using the value of α found in (a), find the distance from point A to the plane Π.

QUESTION 3. **(7 Marks)**

(a) Find the determinant of the matrix $A = \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 1 \\ 0 & a & 1 \end{pmatrix}$, in terms of the unknown constant a. Determine if A is invertible. Justify your answer.

(b) Use part (a) and Cramer's Rule to find the value of x that satisfies the system of linear equations

$$\begin{aligned} 2x + y &= 1 \\ x + z &= 1 \\ 2y + z &= 1. \end{aligned}$$

QUESTION 4. **(8 Marks)**

Consider the function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} e^x \sin x \sin \frac{1}{x} & \text{if } x < 0, \\ 0 & \text{if } x = 0 \\ \dfrac{e^x - 1}{x} & \text{if } x > 0 \end{cases}$$

(a) Does $\lim\limits_{x \to 0^-} f(x)$ exist? If it does, what is its value? If the limit does not exist, explain why.

(b) Is f continuous at $x = 0$? Justify your answer.

QUESTION 5 **(15 Marks)**

(a) Consider the function $f(x) = \begin{cases} ae^x + 1 & \text{if } x < 0, \\ x + b & \text{if } x \geq 0 \end{cases}$ for some constants a, b.

(i) Find $f'(x)$ for $x \neq 0$.

(ii) Determine the values of a and b if f is differentiable on $\mathbb{R}$.

Question 5 continues on Page 7.

QUESTION 5.

(b) Use the definition of derivatives to prove the product rule: If the functions f and g are differentiable at $x = a$, then

$$(fg)'(a) = f(a)\,g'(a) + f'(a)\,g(a).$$

QUESTION 6. **(15 Marks)**

(a) (i) Show that the equation $x^4 - 4x + 1 = 0$, has at least two roots in $[0, 4]$.

(ii) Explain why the equation $x^4 - 4x + 1 = 0$ has exactly two real roots.

Question 6 continues on Page 9.

QUESTION 6.

(b) Use linear approximation to estimate the value of $\sqrt[3]{997}$.

QUESTION 7. **(20 Marks)**

(a) Let $f(x) = \dfrac{x^2}{(x^2-1)}$.

(i) State the domain of f.

(ii) Find the derivative $f'(x)$. Hence determine all interval(s) on which f is increasing.

(iii) Find $f''(x)$ and determine intervals on which the graph of $y = f(x)$ is concave upwards.

(iv) Does $f(x)$ have a global maximum? Justify your answer.

Question 7 continues on Page 11.

(b) Consider rectangles with sides parallel to the axes. Find the area of the largest rectangle that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Give your answers in terms of the positive constants a and b.

QUESTION 8. **(20 Marks)**

(a) Find the area enclosed by the graph of $y = \cos(\ln x)$ and the x-axis, from $x = 1$ to $x = 4$.

Question 8 continues on Page 13.

(b) (i) Show that if f is continuous function, then

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

(ii) Use the result in (i) to show that $\displaystyle\int_0^{\pi/2} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx = \frac{\pi}{4}$.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x - a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{sech } x) = -\text{sech} x \tanh x \qquad \frac{d}{dx}(\text{csch } x) = -\text{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$