Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 9**

Topics: L'Hospital's Rule, Mean Value Theorem, First and second derivatives, Increasing/Decreasing, Concavity, Local extrema, Antiderivatives.

1. Find each of the following limits.
   Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule does not apply, explain why.

   (a) $\displaystyle\lim_{x\to 0} \frac{x+\tan x}{\sin x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - (x^2/2)}{x^3}$

   (c) $\displaystyle\lim_{t\to \infty} \frac{\pi t^5 - 9t^3 + 5}{t^5 + 7t^4 + 3t^2 + -1}$

   (d) $\displaystyle\lim_{x\to -\infty} x^2 e^x$

   (e) $\displaystyle\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

   (f) $\displaystyle\lim_{x\to \infty} \frac{3x + e^{2x}}{x^2 + e^{3x}}$

   (g) $\displaystyle\lim_{x\to \infty} x \tan^{-1}\left(\frac{1}{x}\right)$

   (h) $\displaystyle\lim_{x\to \infty} (e^x + x)^{1/x}$

   (Answers: (a) 2 (b) 1/6 (c) π (d) 0 (e) 1/2 (f) 0 (g) 1 (h) e)

   [Solution]

   (a) $\displaystyle\underbrace{\lim_{x\to 0} \frac{x+\tan x}{\sin x}}_{`\frac{0}{0}'} \stackrel{L'H}{=} \lim_{x\to 0} \frac{1+\sec^2 x}{\cos x} = \frac{1+1}{1} = 2$

   (b)
$$\underbrace{\lim_{x\to 0} \frac{e^x - 1 - x - (x^2/2)}{x^3}}_{`\frac{0}{0}'} \stackrel{L'H}{=} \underbrace{\lim_{x\to 0} \frac{e^x - x - 1}{3x^2}}_{`\frac{0}{0}'} \stackrel{L'H}{=} \underbrace{\lim_{x\to 0} \frac{e^x - 1}{6x}}_{`\frac{0}{0}'} \stackrel{L'H}{=} \lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6}$$

   (c) $\displaystyle\lim_{t\to \infty} \frac{\pi t^5 - 9t^3 + 5}{t^5 + 7t^4 + 3t^2 - 1} = \lim_{t\to \infty} \frac{\pi - \frac{9}{t^2} + \frac{5}{t^5}}{1 + \frac{7}{t} + \frac{3}{t^3} - \frac{1}{t^5}} = \pi$

   (d)
$$\underbrace{\lim_{x\to -\infty} x^2 e^x}_{`(\infty)(0)'} = \underbrace{\lim_{x\to -\infty} \frac{x^2}{e^{-x}}}_{`\frac{\infty}{\infty}'} \stackrel{L'H}{=} \lim_{x\to -\infty} \frac{2x}{-e^{-x}} \stackrel{L'H}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}} = 0$$

(e)

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\underbrace{\lim_{x\to 1}\frac{x-1-\ln x}{(\ln x)\,(x-1)}}_{`\frac{0}{0}'}\overset{L'H}{=}\lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{(x-1)}{x}+\ln x}$$

$$=\underbrace{\lim_{x\to 1}\frac{(x-1)}{(x-1+x\ln x)}}_{\frac{0}{0}}\overset{L'H}{=}\lim_{x\to 1}\frac{1}{1+(\ln x)+1}=\frac{1}{2}$$

(f) $\displaystyle\lim_{x\to\infty}\frac{3x+e^{2x}}{x^2+e^{3x}}=\lim_{x\to\infty}\frac{3+2e^{2x}}{2x+3e^{3x}}=\lim_{x\to\infty}\frac{4e^{2x}}{2+9e^{3x}}=\lim_{x\to\infty}\frac{8e^{2x}}{27e^{3x}}=\lim_{x\to\infty}\frac{8}{27e^{x}}=0$

(g) $\displaystyle\lim_{x\to\infty}x\tan^{-1}\left(\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\tan^{-1}\left(\frac{1}{x}\right)}{1/x}=\lim_{x\to\infty}\frac{\frac{1}{1+(1/x)^2}\cdot\left(\frac{-1}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{1}{1+(1/x)^2}=1$

(h) $\displaystyle\lim_{x\to\infty}(e^x+x)^{1/x}=\lim_{x\to\infty}\exp\left(\ln(e^x+x)^{1/x}\right)=\exp\left(\lim_{x\to\infty}\ln(e^x+x)^{1/x}\right)$

$$=\exp\left(\lim_{x\to\infty}\frac{1}{x}\ln(e^x+x)\right)=\exp\left(\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}\right)=\exp\left(\lim_{x\to\infty}\frac{\frac{e^x+1}{e^x+x}}{1}\right)$$

$$=\exp\left(\lim_{x\to\infty}\frac{e^x+1}{e^x+x}\right)=\exp\left(\lim_{x\to\infty}\frac{e^x}{e^x+1}\right)=\exp\left(\lim_{x\to\infty}\frac{e^x}{e^x}\right)=\exp(1)=e$$

....

2. Prove, by mathematical induction, that, for every non-negative integer n,

$$\lim_{x\to\infty}\frac{x^n}{e^x}=0.$$

[Proof] When $n=0$, we have

$$\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{x^0}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0.$$

Suppose $\displaystyle\lim_{x\to\infty}\frac{x^n}{e^x}=0$ $(*)$ holds for some non-negative n. We shall prove that $\displaystyle\lim_{x\to\infty}\frac{x^{n+1}}{e^x}=0$.
We apply L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{x^{n+1}}{e^x}=\lim_{x\to\infty}\frac{(n+1)x^n}{e^x}=(n+1)\underbrace{\lim_{x\to\infty}\frac{x^n}{e^x}}_{=0\text{ by }(*)}=(n+1)0=0.$$

By mathematical induction, we have

$$\lim_{x\to\infty}\frac{x^n}{e^x}=0.$$

....

3. Suppose that $3 \le f'(x) \le 7$ for all real numbers x. Use the Mean value Theorem to show that

$$15 \le f(9) - f(4) \le 35.$$

[Solution] Note that f is continuous on $[4, 9]$ and differentiable on $(4, 9)$. By the Mean Value Theorem, there exists $c \in (4, 9)$ such that

$$\frac{f(9) - f(4)}{9 - 4} = f'(c)\,, \text{i.e.,} \frac{f(9) - f(4)}{5} = f'(c)\,.$$

Since $3 \le f'(c) \le 7$, it follows that

$$3 \le \frac{f(9) - f(4)}{5} \le 7.$$

Hence, we have

$$15 \le f(9) - f(4) \le 35.$$

....

4. Consider the equation

$$x^3 + 3x^2 + 4x + 1 = 0.$$

(a) Use the Intermediate Value Theorem to show that the above equation has at least one real root.

(b) Use Mean value Theorem to show that the equation $x^3 + 3x^2 + 4x + 1 = 0$ has at most one real root.

(c) Conclude from Parts (a) and (b) that the above equation has exactly one real solution.

[Solution]

(a) Let $f(x) = x^3 + 3x^2 + 4x + 1$, which is continuous on $\mathbb{R}$.
We evaluate $f(x)$ at some values: $f(0) = 1 > 0$ and $f(-1) = -1 < 0$.
Since f is continuous on $[-1, 0]$, $f(0) > 0$ and $f(-1) < 0$, by Intermediate Value Theorem $f(c) = 0$ for some $c \in (-1, 0)$.
Thus, the equation $x^3 + 3x^2 + 4x + 1 = 0$ has at least one real root.

(b) Suppose, on the contrary, that the equation has more than one real roots. That means, there are two distinct real numbers a and b at which $f(a) = 0$ and $f(b) = 0$. We may assume $a < b$.
Note that f is continuous on $[a, b]$ and f is differentiable on (a, b). By Mean Value Theorem, there is a real number $\alpha \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\alpha).$$

However, $f(a) = 0$ and $f(b) = 0$ so that $\dfrac{f(b) - f(a)}{b - a} = 0$. Hence we have $f'(a) = 0$.
The derivative of f is

$$f'(x) = 3x^2 + 6x + 4 = 3(x^2 + 2x) + 4 = 3\left((x + 1)^2 - 1\right) + 4 = 3(x + 1)^2 + 1 \ge 1 > 0.$$

This means that there is no real number α at which $f'(\alpha) = 0$. Therefore, the equation has at most one real root.

(c) From Parts (a) and (b), we conclude that the above equation has exactly one real solution.

....

5. Consider the function $f(x) = 2\sqrt{x} - (3 - \frac{1}{x})$ on $[1, \infty)$.

(a) Explain why f increasing on $[1, \infty)$.

(b) Use part(a) to prove that for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

[SOLUTION]

(a) The function $f(x) = 2\sqrt{x} - (3 - \frac{1}{x})$ is continuous on $[1, \infty)$ since $\sqrt{x}$ and $\frac{1}{x}$ are continuous on $[1, \infty)$.
Differentiating f, gives

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2} = \frac{x\sqrt{x} - 1}{x^2} > 0,$$

for $x > 1$. Therefore, f is increasing on $[1, \infty)$

(b) Use part(a), we have

$$f(x) > f(1), x > 1.$$

That is,

$$2\sqrt{x} - (3 - \frac{1}{x}) > f(1) = 0, x > 1.$$

Therefore, all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

Comment The method used in this question is one standard technique applied to verify an inequality, if the function involved is differentiable.

....

6. Determine the global maximum value of the function $f(x) = \frac{e^x}{1 + e^{2x}}$, $x \in \mathbb{R}$. Justify your answer.

[Solution]

With $f(x) = \frac{e^x}{1 + e^{2x}}$, we have, at every $x \in \mathbb{R}$,

$$f'(x) = \frac{e^x}{e^{2x} + 1} - 2e^x \frac{e^{2x}}{(e^{2x} + 1)^2} = \frac{\left(e^x - e^{3x}\right)}{(e^{2x} + 1)^2} = \frac{e^x \left(1 - e^{2x}\right)}{(e^{2x} + 1)^2}.$$

Note that $f'(x) > 0$ if and only if $(1 - e^{2x}) > 0$, i.e., $e^{2x} < 1$, i.e., $x < 0$.

Hence f is increasing on $(-\infty, 0]$.

Similarly, $f'(x) < 0$ if and only if $(1 - e^{2x}) < 0$, i.e., $e^{2x} > 1$, i.e., $x > 0$. Hence f is decreasing on $[0, \infty)$.

Since f is continuous on $(-\infty, \infty)$, we conclude that f has a global maximum at $x = 0$. The global maximum is $f(0) = 0.5$.

Remark From $f''(0) < 0$, we can only conclude that it is a **local** maximum.

....

7. Classify all critical points of the following functions.

   (a) $f(x) = \sqrt{3 + 2x - x^2}$, for $x \in (-1, 3)$.

   (b) $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}$, for $x \in (0, 2\pi)$.

   (c) $f(x) = x^3 - 2x + 4$ for $x \in \mathbb{R}$.

   [Solution]

   (a) For $f(x) = \sqrt{3 + 2x - x^2}$, where $x \in (-1, 3)$, we have

   $$f'(x) = \frac{2 - 2x}{2\sqrt{3 + 2x - x^2}} = \frac{1 - x}{\sqrt{3 + 2x - x^2}}, x \in (-1, 3).$$

   Note that there is no singular point of f in $(-1, 3)$.
   Setting $f'(x) = 0$, we have $x = 1$, which gives a stationary point of f.
   Thus, $x = 1$ is the only critical point of f in $(-1, 3)$.
   For $-1 < x < 1$, $f'(x) > 0$ whereas $f'(x) < 0$ for $1 < x < 3$.
   By the first derivative test, we conclude that $f(1)$ is a local maximum.

   Note The second derivative of this function is quite involved to evaluate. Thus, it is better to use first derivative test.

   (b) For $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}$, where $x \in (0, 2\pi)$, we have

   $$f'(x) = \frac{1}{2} - \cos\frac{x}{2}, \text{ and } f''(x) = \frac{1}{2}\sin\frac{x}{2}, x \in (0, 2\pi).$$

   Note that $f'(x) = 0$ if and only if $\frac{x}{2} = \frac{\pi}{3}$, i.e., $x = \frac{2\pi}{3}$.
   We have $f''(\frac{2\pi}{3}) = \frac{1}{2}\sin\frac{2\pi}{3} > 0$.
   By second derivative test, we conclude that $f(\frac{2\pi}{3})$ is a local minimum.

   (c) Consider $f(x) = x^3 - 2x + 4$ for $x \in \mathbb{R}$. We have

   $$f'(x) = 3x^2 - 2 \text{ and } f''(x) = 6x, x \in \mathbb{R}.$$

   Note that there is no singular point and $f'(x) = 0$ for $x = \pm\sqrt{\frac{2}{3}}$.
   Now, we have $f''\left(\sqrt{\frac{2}{3}}\right) = 6\sqrt{\frac{2}{3}} > 0$, and $f''\left(-\sqrt{\frac{2}{3}}\right) = -6\sqrt{\frac{2}{3}} < 0$.
   By second derivative test, we conclude that $f\left(\sqrt{\frac{2}{3}}\right)$ is a local minimum whereas $f\left(-\sqrt{\frac{2}{3}}\right)$ is a local maximum.

....

8. Find the general antiderivative for each of the following functions. Check your answers by differentiation.

   (a) $\sec^2 2x - \sin(3x+5)$

   (b) $\left(1-x^2\right)^2 + \dfrac{1}{1+3x}$

   (c) $e^{2x} + \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{x^2+1}$

   [Solution]

   (a) $\displaystyle\int \sec^2 2x - \sin(3x+5)\ dx = \frac{1}{2}\tan(2x) + \frac{1}{3}\cos(3x+5) + C$

   (b) $\displaystyle\int \left(1-x^2\right)^2 + \frac{1}{1+3x}\ dx = \int \left(1-2x^2+x^4\right) + \frac{1}{1+3x}\ dx = x - \frac{2x^3}{3} + \frac{x^5}{5} + \frac{1}{3}\ln|1+3x| + C)$

   (c) $\displaystyle\int e^{2x} + \frac{1}{\sqrt{1-x^2}} - \frac{1}{x^2+1}\ dx = \frac{e^{2x}}{2} + \sin^{-1}(x) - \tan^{-1}x + C$

. . . .

9. Find the following indefinite integrals. Check your answers by differentiation.

   (a) $\displaystyle\int (\cos 2x + 2\cos x)\ dx$

   (b) $\displaystyle\int \left(1+\tan^2\theta\right)\ d\theta$

   (c) $\displaystyle\int \cot^2 x + 3\sec^2(3x)\ dx$

   [Solution]

   (a) $\displaystyle\int (\cos 2x + 2\cos x)\ dx = \frac{\sin 2x}{2} + 2\sin x + C$

   (b) $\displaystyle\int \left(1+\tan^2\theta\right)\ d\theta = \int \left(\sec^2\theta\right)\ d\theta = \tan\theta + C$

   (c) $\displaystyle\int \cot^2 x + 3\sec^2(3x)\ dx = \int \left(\csc^2 x - 1\right) + 3\sec^2(3x)\ dx = -\cot x - x + \tan 3x + C$

. . . .

10. Find the curve $y = f(x)$ that passes through the point $(9,4)$ and whose gradient at each point (x,y) is $3\sqrt{x}$.

    [Solution] Note that

$$f'(x) = \frac{dy}{dx} = 3\sqrt{x} = 3x^{1/2},$$

    so that

$$f(x) = \frac{3x^{3/2}}{3/2} + C = 2x^{3/2} + C.$$

    When $x = 9$, $y = 4$ which gives

$$4 = 2(27) + C \Rightarrow C = -50$$

    Thus,

$$y = 2x^{3/2} - 50.$$

. . . .

(a) Express $\left(\dfrac{2k-n}{n^2}\right)$ as $\dfrac{1}{n}\ f(\dfrac{k}{n})$ for some function f.

(b) Using part (a) and $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{1}{n}\ f\left(\dfrac{k}{n}\right)=\int_0^1 f(x)\ dx$, express the limit $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{2k-n}{n^2}$ as the definite integral $\int_0^1 f(x)\ dx$ and use it to evaluate the limit.

[Solution]

(a) Note that $\left(\dfrac{2k-n}{n^2}\right)=\dfrac{1}{n}\left(\dfrac{2k-n}{n}\right)=\dfrac{1}{n}\left(2(\dfrac{k}{n})-1\right)$.
Thus, we have $f(x)=2x-1$. (Replacing $\frac{k}{n}$ by x)

(b) Using part (a) and $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{1}{n}\ f\left(\dfrac{k}{n}\right)=\int_0^1 f(x)\ dx$, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2k-n}{n^2}$$

$$=\underbrace{\lim_{n\to\infty}\sum_{k=1}^{n}}_{\int_0^1}\underbrace{\frac{1}{n}}_{\text{`}dx\text{'}}\underbrace{\left(2\left(\frac{k}{n}\right)-1\right)}_{f(\frac{k}{n})}.$$

$$=\int_0^1 (2x-1)\ dx=\left[x^2-x\right]_0^1=0.$$

. . . .

11. Express each of the following limits as a definite integral $\int_0^1 f(x)\ dx$ and use it to evaluate the limit.

(a) $\lim\limits_{n\to\infty}\dfrac{1}{n}\left\{\sin(\dfrac{\pi}{n})+\sin(\dfrac{2\pi}{n})+\cdots+\sin(\dfrac{k\pi}{n})+\cdots+\sin(\dfrac{(n-1)\pi}{n})+\sin(\dfrac{n\pi}{n})\right\}$

(b) $\lim\limits_{n\to\infty}\left\{\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+k}+\cdots+\dfrac{1}{n+n}\right\}$

(c) $\lim\limits_{n\to\infty}\dfrac{1^2+2^2+\cdots+n^2}{n^3}$

(Answers: (a) $\frac{2}{\pi}$ (b) $\ln 2$ (c) $\frac{1}{3}$.)
[Solution]

(a) $\lim\limits_{n\to\infty}\dfrac{1}{n}\left\{\sin(\dfrac{\pi}{n})+\sin(\dfrac{2\pi}{n})+\cdots+\sin(\dfrac{k\pi}{n})+\cdots+\sin(\dfrac{(n-1)\pi}{n})+\sin(\dfrac{n\pi}{n})\right\}$

$$=\lim_{n\to\infty}\sum_{1}^{n}\frac{1}{n}\sin(\frac{k\pi}{n})=\lim_{n\to\infty}\sum_{1}^{n}\frac{1}{n}\underbrace{\sin(\frac{k}{n}\pi)}_{f(\frac{k}{n})}$$

$$=\int_0^1\sin(x\pi)\ dx=\left[\frac{-\cos(x\pi)}{\pi}\right]_0^1=\frac{-\cos(\pi)+\cos(0)}{\pi}=\frac{2}{\pi}$$

(b) Rewriting

$$\frac{1}{n+k} = \frac{1}{n}\left(\frac{1}{1+\frac{k}{n}}\right),$$

we have

$$\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+k}+\cdots+\frac{1}{n+n}\right\}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\ \frac{1}{n+k}$$

$$= \underbrace{\lim_{n\to\infty}\sum_{k=1}^{n}}_{\int_0^1}\ \underbrace{(\frac{1}{n})}_{\text{`}dx\text{'}}\underbrace{\left(\frac{1}{1+\frac{k}{n}}\right)}_{f(x)}$$

$$= \int_0^1 \left(\frac{1}{1+x}\right)\ dx = [\ln|1+x|]_0^1 = \ln 2.$$

(c) $\displaystyle \lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}$

$$= \lim_{n\to\infty}\frac{1^2+2^2+\cdots+k^2+\cdots+n^2}{n^3}$$

$$= \lim_{n\to\infty}\left(\frac{1^2}{n^3}+\frac{2^2}{n^3}+\cdots\frac{k^2}{n^3}+\cdots+\frac{n^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^2$$

$$= \int_0^1\ x^2\ dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$