Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 9**

Topics: L'Hospital's Rule, Mean Value Theorem, First and second derivatives, Increasing/Decreasing, Concavity, Local extrema, Antiderivatives, indefinite and definite integrals.

1. Find each of the following limits.
   Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule does not apply, explain why.

   (a) $\displaystyle\lim_{x\to 0} \frac{x + \tan x}{\sin x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - (x^2/2)}{x^3}$

   (c) $\displaystyle\lim_{t\to \infty} \frac{\pi t^5 - 9t^3 + 5}{t^5 + 7t^4 + 3t^2 - 1}$

   (d) $\displaystyle\lim_{x\to -\infty} x^2 e^x$

   (e) $\displaystyle\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

   (f) $\displaystyle\lim_{x\to \infty} \frac{3x + e^{2x}}{x^2 + e^{3x}}$

   (g) $\displaystyle\lim_{x\to \infty} x \tan^{-1}\left(\frac{1}{x}\right)$

   (h) $\displaystyle\lim_{x\to \infty} (e^x + x)^{1/x}$

2. Prove, by mathematical induction, that, for every positive integer n,

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

3. Suppose that $3 \le f'(x) \le 7$ for all real numbers x. Use the Mean value Theorem to show that

$$15 \le f(9) - f(4) \le 35.$$

4. Consider the equation

$$x^3 + 3x^2 + 4x + 1 = 0.$$

   (a) Use the Intermediate Value Theorem to show that the above equation has at least one real root.

   (b) Use Mean value Theorem to show that the equation $x^3 + 3x^2 + 4x + 1 = 0$ has at most one real root.

   (c) Conclude from Parts (a) and (b) that the above equation has exactly one real solution.

5. Consider the function $f(x) = 2\sqrt{x} - (3 - \frac{1}{x})$ on $[1, \infty)$.

   (a) Explain why f increasing on $[1, \infty)$.

   (b) Use part(a) to prove that for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

6. Determine the global maximum value of $f(x) = \dfrac{e^x}{1 + e^{2x}}$, $x \in \mathbb{R}$. Justify your answer.

7. Classify all critical points of the following functions.

   (a) $f(x) = \sqrt{3 + 2x - x^2}$, for $x \in (-1, 3)$.

   (b) $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}$, for $x \in (0, 2\pi)$.

   (c) $f(x) = x^3 - 2x + 4$ for $x \in \mathbb{R}$.

8. Find the general antiderivative for each of the following functions. Check your answers by differentiation.

   (a) $\sec^2 2x - \sin(3x + 5)$

   (b) $\left(1 - x^2\right)^2 + \dfrac{1}{1 + 3x}$

   (c) $e^{2x} + \dfrac{1}{\sqrt{1 - x^2}} - \dfrac{1}{x^2 + 1}$

9. Find the following indefinite integrals. Check your answers by differentiation.

   (a) $\displaystyle\int (\cos 2x + 2\cos x)\ dx$

   (b) $\displaystyle\int \left(1 + \tan^2\theta\right)\ d\theta$

   (c) $\displaystyle\int \cot^2 x + 3\sec^2(3x)\ dx$

10. Find the curve $y = f(x)$ that passes through the point $(9, 4)$ and whose gradient at each point (x, y) is $3\sqrt{x}$.

11. (a) Express $\left(\dfrac{2k - n}{n^2}\right)$ as $\dfrac{1}{n}\ f(\dfrac{k}{n})$ for some function f.

    (b) Using part (a) and $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\ f\left(\frac{k}{n}\right) = \int_0^1 f(x)\ dx$, express the limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2k - n}{n^2}$ as the definite integral $\displaystyle\int_0^1 f(x)\ dx$ and use it to evaluate the limit.

12. Express each of the following limits as a definite integral $\displaystyle\int_0^1 f(x)\ dx$ and use it to evaluate the limit.

    (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left\{\sin(\frac{\pi}{n}) + \sin(\frac{2\pi}{n}) + \cdots + \sin(\frac{k\pi}{n}) + \cdots + \sin(\frac{(n-1)\pi}{n}) + \sin(\frac{n\pi}{n})\right\}$

    (b) $\displaystyle\lim_{n\to\infty} \left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+k} + \cdots + \frac{1}{n+n}\right\}$

    (c) $\displaystyle\lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3}$

Answer

1. (a) 2
   (b) 1/6
   (c) π
   (d) 0
   (e) 1/2
   (f) 0
   (g) 1
   (h) e

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6. The global maximum is $f(0) = 0.5$.

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7. (a) $f(1)$ is a local maximum.
   (b) $f(\frac{2\pi}{3})$ is a local minimum.
   (c) $f\left(\sqrt{\frac{2}{3}}\right)$ is a local minimum whereas $f\left(-\sqrt{\frac{2}{3}}\right)$ is a local maximum.

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8. (a) $\frac{1}{2}\tan(2x) + \frac{1}{3}\cos(3x+5) + C$
   (b) $x - \frac{2x^3}{3} + \frac{x^5}{5} + \frac{1}{3}\ln|1+3x| + C$
   (c) $\frac{e^{2x}}{2} + \sin^{-1}(x) - \tan^{-1}x + C$

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9. (a) $\frac{\sin 2x}{2} + 2\sin x + C$
   (b) $\tan\theta + C$
   (c) $-\cot x - x + \tan 3x + C$

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10. The curve is $y = 2x^{3/2} - 50$.

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11. (a) $f(x) = 2x - 1$
    (b) $\int_0^1 (2x-1)\ dx = 0$.

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12. (a) $\frac{2}{\pi}$
    (b) $\ln 2$
    (c) $\frac{1}{3}$