Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 4**

1. For each of the following matrices, find (i) its cofactor matrix, (ii) its Adjoint matrix and (iii) the product of the matrix and its adjoint matrix.

   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

   (b) $B = \begin{pmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

   Answers

   (a) (i) $\begin{pmatrix} 2 & -3 \\ -2 & 1 \end{pmatrix}$, (ii) $\mathrm{adj}(A) = \begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix}$, (iii) $= \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$

   (b) (i) $\begin{pmatrix} 0 & 4 & -8 \\ -5 & 1 & 3 \\ 5 & -1 & 17 \end{pmatrix}$ (ii) $\mathrm{adj}(B) = \begin{pmatrix} 0 & -5 & 5 \\ 4 & 1 & -1 \\ -8 & 3 & 17 \end{pmatrix}$, (iii) $\begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix}$

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2. Evaluate the following determinant by inspection.

   (a) $\begin{vmatrix} 3 & -17 & -3 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -30$ (upper triangular matrix)

   (b) $\begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ -8 & \sqrt{2} & 0 & 0 \\ 7 & 0 & -1 & 0 \\ 9 & 5 & 1 & 6 \end{vmatrix} = -12$ (lower triangular matrix)

   (c) $\begin{vmatrix} 1 & -4 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 9 & 0 & -7 & 0 \\ -11 & 3 & 0 & 1 \end{vmatrix} = 0$ (zero row)

   (d) $\begin{vmatrix} 1 & 7 & 9 \\ \sqrt{2} & \pi & e \\ 1 & 7 & 9 \end{vmatrix} = 0$ (Identical rows)

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3. Let $A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

(a) (i) $C_{21} = -\begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 6$ (ii) $C_{23} = -\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = -12$

(iii) $C_{44} = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 0$ (iv) $C_{13} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 6$

(b) Evaluate the determinant of A by cofactor expansion along

(i) the first column,

$$|A| = 2\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 2(-4) + 6 = -2$$

(ii) the third row.

$$|A| = -2\begin{vmatrix} 2 & 3 & 3 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix} = -2(-1) + 2(-2) = -2$$

Exercise Evaluate $|A|$ by cofactor expansion along second column.

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4. Solve for all real numbers x which satisfies the following equation.

$$\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$$

[Solution] Evaluating determinants of matrices on both sides of the equation, we have

$$x(1-x) + 3 = (x(x-5) + 18) - 3\,(6-x)\,.$$

Solving for x, (we have omitted the detail here), we have $x = \dfrac{3 \pm \sqrt{33}}{4}$.

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5. For the matrix $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$, find A^{-1} using the following formula $A^{-1} = \dfrac{1}{\det(A)}\text{adj}(A)$.

Answer: $\det(A) = -6$ and $A^{-1} = \begin{bmatrix} 2 & 0 & 3/2 \\ 2/3 & 1/3 & 2/3 \\ -1 & 0 & -1 \end{bmatrix}$

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6. The matrix $R = \begin{bmatrix} \cos\theta & \sin\,\theta & 0 \\ -\sin\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of rotation of points in $\mathbb{R}^3$, it rotates points about the z-axis by θ radians in counter-clockwise direction.

Show that the matrix R invertible for all values of θ and find the inverse R^{-1} of R.

Solution Note that R is invertible if and only if $|R| \neq 0$.

Evaluating the determinant of R by cofactors along the third row, we have

$$|R| = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Therefore, R is invertible.

Using the formula to find $R^{-1} = \frac{1}{\det(R)}\text{adj}(R)$, we have

$$R^{-1} = \begin{bmatrix} \cos\theta & -\sin\,\theta & 0 \\ \sin\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark (for students who are keen to know.)

We have seen in the previous tutorial that the matrix of rotation in 2-dimensional space, about the origin and θ radians in counter-clockwise direction, is $\begin{bmatrix} \cos\theta & \sin\,\theta \\ -\sin\,\theta & \cos\theta \end{bmatrix}$, and the rotated point has coordinates $(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.

To see that R represents rotation about the z-axis, we evaluate the matrix product:

$$R\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\,\theta & 0 \\ -\sin\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{pmatrix}$$

The z-component is unchanged. The first two coordinates indicate rotated point in the xy-plane.

Intuitively the inverse of R is the matrix of rotation of points about z-axis by θ radians clockwise direction. Indeed, rewriting the matrix R^{-1} using $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, we obtain

$$R^{-1} = \begin{bmatrix} \cos(-\theta) & \sin\,(-\theta) & 0 \\ -\sin\,(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

which is the matrix of rotation about the z-axis by $(-\theta)$ radians in the counter-clockwise direction, which is θ radians in the clockwise direction.

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7. Solve the linear system by Cramer's rule, if it applies.

$$\begin{array}{rcrcrcr} 4x & + & 5y & & & = & 2 \\ 11x & + & y & + & 2z & = & 3 \\ x & + & 5y & + & 2z & = & 1 \end{array}$$

[SOLUTION] Let $A = \begin{pmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{pmatrix}$. Note that $|A| = -132 \neq 0$ so that we can apply Cramer's Rule.

$$A_1 = \begin{pmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{pmatrix}$$

We have $|A_1| = -36, |A_2| = -24, |A_3| = 12$. By Cramer's Rule we have

$$x = \frac{|A_1|}{|A|} = \frac{3}{11}, y = \frac{|A_1|}{|A|} = \frac{2}{11}, z = \frac{|A_1|}{|A|} = -\frac{1}{11}.$$

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8. Solve for x, y and z.

$$\begin{array}{rcrcrcr} \frac{1}{x} & + & \frac{2}{y} & + & \frac{1}{z} & = & 1 \\ \frac{3}{x} & + & \frac{4}{y} & + & \frac{1}{z} & = & 5 \\ \frac{8}{x} & + & \frac{6}{y} & + & \frac{7}{z} & = & 0 \end{array}$$

Solution Rewriting the above as a matrix equation, we have

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 8 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1/x \\ 1/y \\ 1/z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}.$$

Note that $|A| = -18 \neq 0$ so that Cramer's Rule is applicable. We have

$$|A_1| = \begin{vmatrix} \mathbf{1} & 2 & 1 \\ \mathbf{5} & 4 & 1 \\ \mathbf{0} & 6 & 7 \end{vmatrix} = -18, \ |A_2| = \begin{vmatrix} 1 & \mathbf{1} & 1 \\ 3 & \mathbf{5} & 1 \\ 8 & \mathbf{0} & 7 \end{vmatrix} = -18, \ |A_3| = \begin{vmatrix} 1 & 2 & \mathbf{1} \\ 3 & 4 & \mathbf{5} \\ 8 & 6 & \mathbf{0} \end{vmatrix} = 36.$$

By Cramer's Rule, we have

$$\frac{1}{x} = \frac{|A_1|}{|A|} = 1, \frac{1}{y} = \frac{|A_2|}{|A|} = 1 \ \ \& \ \ \frac{1}{z} = \frac{|A_3|}{|A|} = -2.$$

Hence $x = y = 1$ and $z = -1/2$.

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9. (AY 2012/13 Semester 1) Consider the following system of linear equations

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & c & = & 1 \\ -a & + & 4b & + & 2c & = & 0 \\ a & + & rb & - & c & = & -1 \end{array}$$

(i) Find the values of r at which Cramer's rule is applicable.

(ii) For $r = 1$, use Cramer's Rule to determine the unknown b.

Solution

(i) Let $A = \begin{pmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 1 & r & -1 \end{pmatrix}$

$$\det(A) = 2\begin{vmatrix} 4 & 2 \\ r & -1 \end{vmatrix} - 3\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ 1 & r \end{vmatrix} = -1 - 3r$$

Cramer's Rule is applicable if and only if $\det(A) \neq 0$, i.e., $r \neq -\frac{1}{3}$.

(ii) Put $r = 1$.

$\det(A) = -1 - 3 = -4$

$\det(A_2) = \det\begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix} = -\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4$

Therefore $b = \frac{4}{-4} = -1$.

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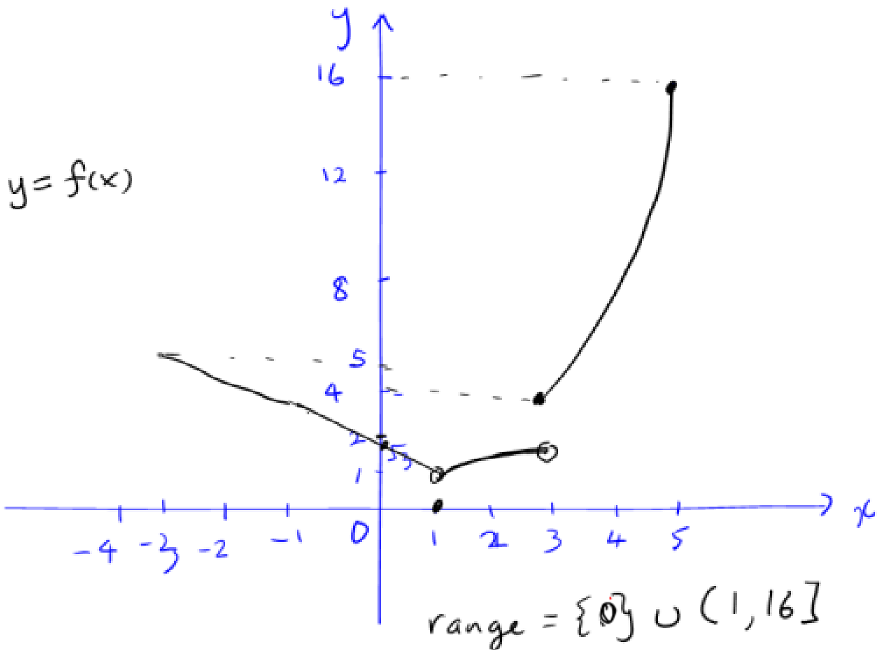
10. Consider the function $f : [-3, 5] \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} 2 - x & \text{if } -3 \le x < 1 \\ 0 & \text{if } x = 1 \\ \sqrt{x} & \text{if } 1 < x < 3 \\ (x-1)^2 & \text{if } 3 \le x \le 5. \end{cases}$$

(a) Sketch the graph $y = f(x)$ for $-3 \le x \le 5$. From your sketch, write down the range of f, i.e., the set of values where $f(x)$ assumes for $-3 \le x \le 5$.

(b) From your graph, determine each of the following limits if it exists:
(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to 2} f(x)$ (iii) $\lim_{x\to 4} f(x)$ (iv) $\lim_{x\to 1^-} f(x)$ (v) $\lim_{x\to 1^+} f(x)$
(vi) $\lim_{x\to 1} f(x)$ (vii) $\lim_{x\to 3} f(x)$

(a) This type of function is called piecewise function or piecewisely defined function.
Guidelines: To sketch the graph of $y = f(x)$, we proceed piece-by-piece as follows:
For $-3 \le x < 1$, note that $f(x) = 2 - x$. We sketch the graph of $y = 2 - x$ using dotted curves, and use only the piece for $x \in [-3, 1)$. A solid dot at $x = -3$ to indicate $x = -3$ is inclusive and a hollow one at $x = 1$ for excluding $x = 1$.
Similarly for the remaining pieces.



From the graph, we have the range of f is $\{0\} \cup (1, 16]$.

(b) From the graph, we have

(i) $\lim_{x\to 0} f(x) = 2$

(ii) $\lim_{x\to 2} f(x) = \sqrt{2}$

(iii) $\lim_{x\to 4} f(x) = 9$

(iv) $\lim_{x\to 1^-} f(x) = 1$

(v) $\lim_{x\to 1^+} f(x) = 1$

(vi) $\lim_{x\to 1} f(x) = 1$. Note that $\lim_{x\to 1} f(x) \neq 0$ where $f(1) = 0$.

(vii) $\lim_{x \to 3} f(x)$ does not exist, since $\lim_{x \to 3^-} f(x) = \sqrt{3} \neq \lim_{x \to 3^+} f(x) = 4$.

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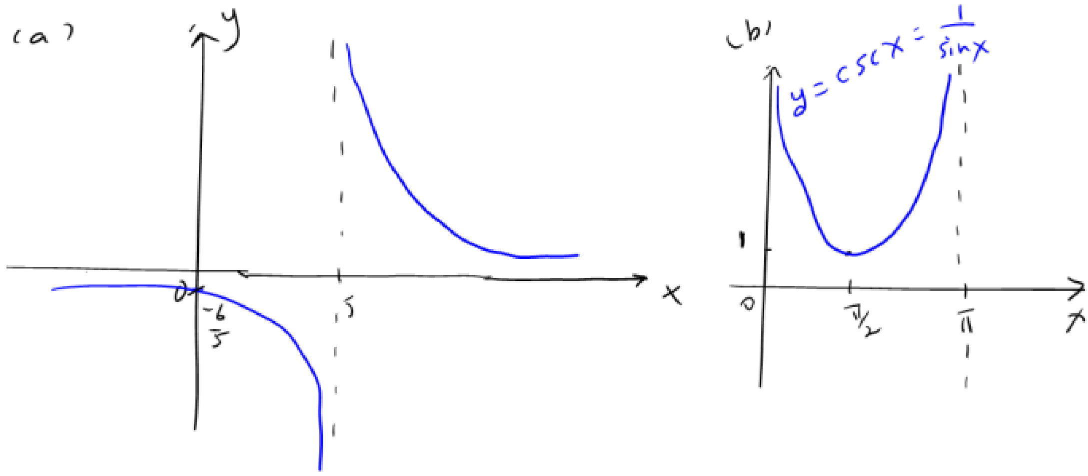
11. Does the following limit exist? If it does, what is its value? If it is an infinite limit, determine whether it is $+\infty$ and $-\infty$.

(a) $\lim_{x \to 5^+} \dfrac{6}{x-5} = +\infty$

(b) $\lim_{x \to \pi^-} \csc x = \lim_{x \to \pi^-} \dfrac{1}{\sin x} = +\infty$. Note that $\sin x > 0$ for $\pi/2 < x < \pi$ and $\lim_{x \to \pi^-} \sin x = 0$.



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12. (a) Sketch graphs of exponential functions $y = a^x$, where $0 < a < 1$ and $a > 1$.

(b) Use the graphs in part (a) to write down each of the following limits.

(i) $\lim_{x \to \infty} (1.001)^x = +\infty$

(ii) $\lim_{x \to -\infty} \pi^x = 0$

(iii) $\lim_{x \to \infty} 0.37^x = 0$

(iv) $\lim_{x \to -\infty} 181^x = 0$

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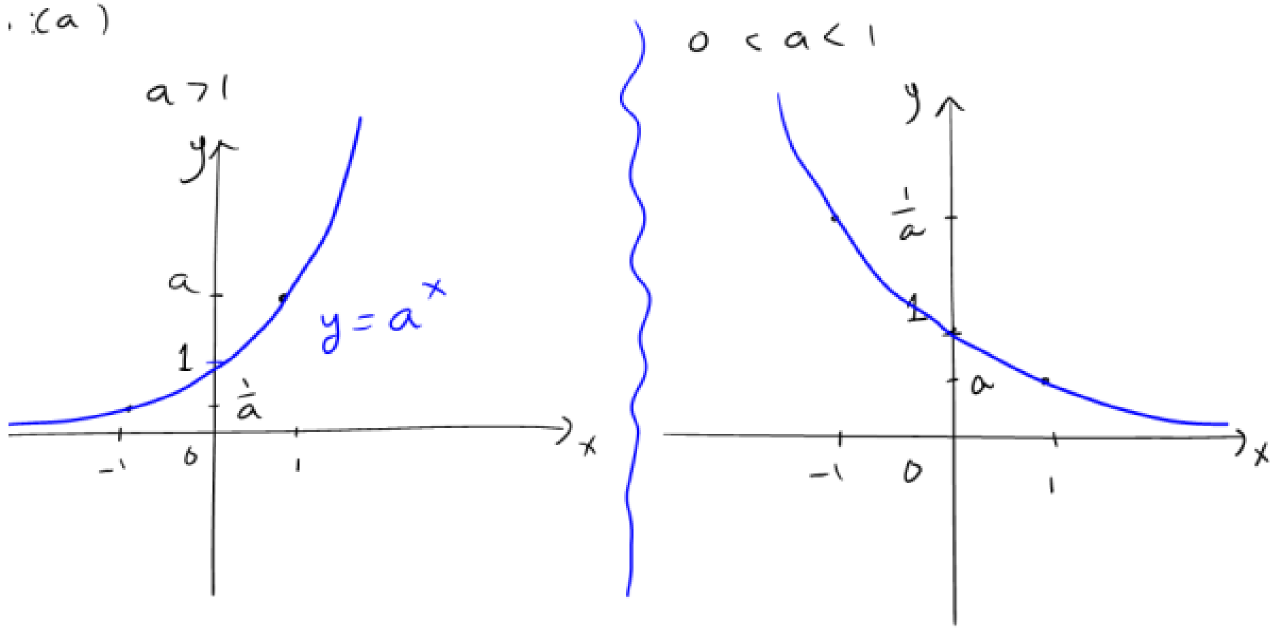

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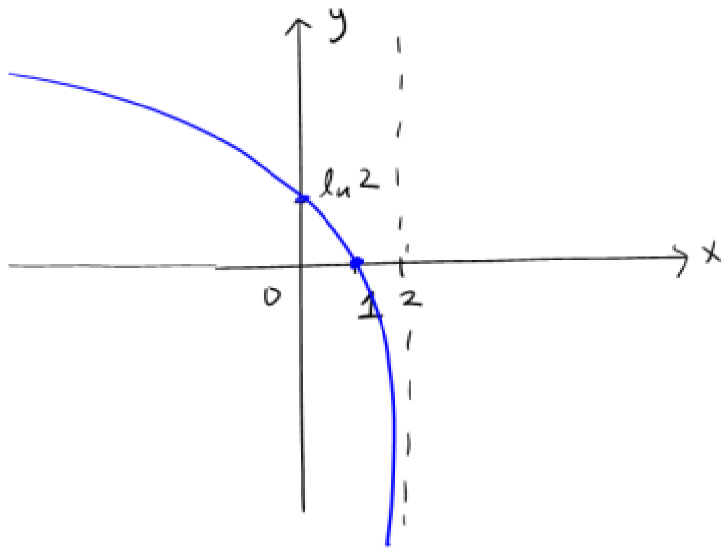
13. Sketch the graph of $y = \ln(2-x)$ and use it to determine each of the following limits.

$\therefore\ y = \ln(2-x)$, for $x < 2$.



(a) $\lim\limits_{x\to 2^-} \ln(2-x) = -\infty$

(b) $\lim\limits_{x\to 1^-} \ln(2-x) = 0$

(c) $\lim\limits_{x\to 3^+} \ln(2-x)$ is not defined since $\ln(2-x)$ is defined for $x < 2$

(d) $\lim\limits_{x\to -3} \ln(2-x) = \ln 5$

(e) $\lim\limits_{x\to -\infty} \ln(2-x) = \infty$

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