Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 1**

1. Evaluate the expression and write your answer in the form $a + bi$, $a, b \in \mathbb{R}$.

   (a) $i^{179} + i^{2013} = i^{4(44)+3} + i^{4(503)+1} = i^3 + i = -i + i = 0$

   (b) $\dfrac{1+2i}{3-4i} = \dfrac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \dfrac{(3-8)+i(6+4)}{3^2+4^2} = \dfrac{-5}{25} + \dfrac{10}{25}i = \dfrac{-1}{5} + \dfrac{2}{5}i$

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2. For each of the following, represent the complex number on the Argand diagram. Find the modulus and the principal argument of the complex number. Hence express the complex number in its polar representation.

   (a) $1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

   (b) $-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

   (c) $1 - \sqrt{3}i = 2\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$

   (d) $-1 - \sqrt{3}i = 2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$

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3. Find the complex conjugate of each of the following complex numbers.

   (a) $2i$ (Answer: $-2i$)

   (b) 2 (Answer: 2)

   (c) $1 + 3i$ (Answer: $1 - 3i$)

   (d) $-3 - 4i$ (Answer: $-3 + 4i$)

   (e) a complex number with modulus 2 and argument $\theta = \frac{\pi}{3}$.
   (Answer: $2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 1 - \sqrt{3}i$)

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4. Sketch the regions defined by

   (a) $\mathrm{Re}(z) \geq 0$

   (b) $\mathrm{Im}(z) < 2$

   (c) $|z| \geq 2$

   (d) $\dfrac{\pi}{6} \leq \arg(z) \leq \dfrac{\pi}{3}$

   (e) $|z - i| = |z - 1|$

   (f) $|z - (1 + i)| \leq 2$

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5. Solve the following equation for the real numbers, x and y.

$$\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1+i$$

[Solution] Note that

$$\frac{1}{x+iy} = 1+i-\left(\frac{1+i}{1-i}\right)^2 = 1+i-\left(\frac{(1+i)(1+i)}{1^2+(-1)^2}\right)^2 = 1+i-(i)^2 = 2+i$$

Therefore,

$$x+iy = \frac{1}{2+i} = \frac{2-i}{5} = \frac{2}{5} + \frac{-1}{5}i.$$

We have $x = \frac{2}{5}$ and $y = \frac{-1}{5}$.

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6. Suppose a complex number $z = x+iy$ satisfies

$$|z-1| = \frac{1}{2}|z-i|.$$

Show that x and y satisfy the following equation

$$\left(x-\frac{4}{3}\right)^2 + \left(y+\frac{1}{3}\right)^2 = \frac{8}{9},$$

which represents a circle of radius $\dfrac{\sqrt{8}}{3}$ centred at $\left(\dfrac{4}{3}, \dfrac{-1}{3}\right)$.

[Solution] From

$$|z-1| = \frac{1}{2}|z-i|,$$

we have

$$|x-1+iy| = \frac{1}{2}|x+i(y-1)|,$$

which is equivalent to

$$|x-1+iy|^2 = \frac{1}{4}|x+i(y-1)|^2.$$

Thus, we have

$$\begin{aligned}
(x-1)^2+y^2 &= \frac{1}{4}\left(x^2+(y-1)^2\right),\\
\Longleftrightarrow x^2-2x+1+y^2 &= \frac{1}{4}\left(x^2+y^2-2y+1\right)\\
\Longleftrightarrow 3x^2-8x+4 &= -3y^2-2y+1\\
\Longleftrightarrow x^2-\frac{8}{3}x+y^2+\frac{2}{3}y+1 &= 0.
\end{aligned}$$

Completing squares $x^2-\frac{8}{3}x = \left(x-\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^2$, and $y^2+\frac{2}{3}y = \left(y+\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2$, we have

$$\begin{aligned}
\left(x-\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^2+\left(y+\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2+1 &= 0,\\
\Longleftrightarrow \left(x-\frac{4}{3}\right)^2+\left(y+\frac{1}{3}\right)^2 &= \frac{8}{9}.
\end{aligned}$$

The last equation is the equation of a circle, with centre $\left(\dfrac{4}{3}, \dfrac{-1}{3}\right)$ and radius $\dfrac{\sqrt{8}}{3}$.

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7. Express each of the following complex numbers in the form $re^{i\theta}$, with $r \geq 0$, and $-\pi < \theta \leq \pi$.

(a) $\left(1+\sqrt{-3}\right)^2$ (b) $\dfrac{1+i}{1-i}$

[Solution]

(a) $\left(1+\sqrt{-3}\right)^2 = -2+2\sqrt{3}i$.

Thus $r = \sqrt{(-2)^2+(2\sqrt{3})^2} = 4$ and $\tan\theta = \frac{\sqrt{3}}{-1}$. Since the complex number lies on second quadrant, we have $\theta = \frac{2\pi}{3}$.

Alternatively, we have $1+\sqrt{-3} = 2e^{\frac{\pi i}{3}}$, thus we have

$$\left(1+\sqrt{-3}\right)^2 = 2^2 e^{\frac{2\pi i}{3}} = 4e^{\frac{2\pi i}{3}}.$$

(b) $\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{2} = i = e^{\frac{\pi i}{2}}$

Alternatively, we have $\dfrac{1+i}{1-i} = \dfrac{\sqrt{2}e^{\frac{\pi i}{4}}}{\sqrt{2}e^{\frac{-\pi i}{4}}} = e^{\frac{\pi i}{4}-\frac{-\pi i}{4}} = e^{\frac{\pi i}{2}}$.

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8. Let $z = a+ib$, where a and b are some real numbers.

Show that

(a) $z+\bar{z} = 2\mathrm{Re}(z)$.

(b) $z = \bar{z} \iff z$ is a real number.

(c) If z is a root of $ax^3+bx^2+cx+d=0$, where a, b, c and d are real constants, then $\bar{z}$ is also a root of $ax^3+bx^2+cx+d=0$.

Remark *More generally, we have*
Suppose that $p(x) = a_0 + a_1x + \ldots + a_nx^n$ is a polynomial in x with real coefficients a_k's. If a complex number z is a solution of $p(x) = 0$, then the conjugate $\bar{z}$ of z is also a solution of $p(x) = 0$.

[Solution]

(a) $z+\bar{z} = (a+ib)+(a-ib) = 2a = 2\mathrm{Re}(z)$.

(b) Let $z = a+ib$. Then

$$z = \bar{z} \iff (a+ib) = (a-ib) \iff b = -b \iff b = 0 \iff z \text{ is a real number.}$$

(c) If z is a root of $ax^3+bx^2+cx+d=0$, where a, b, c and d are real constants, then

$$az^3+bz^2+cz+d = 0 - -(*).$$

(Our aim is to prove $a\bar{z}^3+b\bar{z}^2+c\bar{z}+d=0$.)

Taking conjugation of the equation (*), we have

$$\overline{az^3+bz^2+cz+d} = \bar{0} = 0.$$

Applying properties $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$ and $\overline{z_1\cdot z_2} = \bar{z}_1\cdot\bar{z}_2$, we have

$$\bar{a}\bar{z}^3+\bar{b}\bar{z}^2+\bar{c}\bar{z}+\bar{d} = 0.$$

For each real number a, b c and d, its conjugation is the real number itself. Thus we have

$$a\bar{z}^3+b\bar{z}^2+c\bar{z}+d = 0.$$

The above simply means that $\bar{z}$ is a root of $ax^3+bx^2+cx+d=0$.

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9. Express each of following complex numbers in the form (i) $\cos\alpha + i\sin\alpha$ and (ii) $x + iy$.

(a) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^7 = \cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

(b) $\frac{1}{\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^2} = \left(\frac{1}{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}}\right)^2 = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^2 = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

(c) $\frac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^4} = \frac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^4} = \frac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)}$

$= \text{cis}\left(\frac{\pi}{6} - \frac{8\pi}{3}\right) = \text{cis}\left(\frac{-5\pi}{2}\right) = -i$

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10. Without using any series expansions, prove that

$$(\sqrt{3} + i)^n + (\sqrt{3} - i)^n, \text{ is real.}$$

Find the value of this expression when $n = 12$.

[Solution] Note that $\sqrt{3} + i = 2 \text{ cis } (\frac{\pi}{6})$ and $\sqrt{3} - i = 2 \text{ cis } (\frac{-\pi}{6})$. By De Moivre's Theorem, we have

$$\left(\sqrt{3} \pm i\right)^n = 2^n \text{ cis } (\frac{\pm n\pi}{6}).$$

Therefore, we have

$$(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = \left(2 \text{ cis } (\frac{\pi}{6})\right)^n + \left(2 \text{ cis } (\frac{-\pi}{6})\right)^n$$

$$= \left(2^n \text{ cis } (\frac{n\pi}{6})\right) + \left(2^n \text{ cis } (\frac{-n\pi}{6})\right) = 2^n \left((\cos(\frac{n\pi}{6}) + i\sin(\frac{n\pi}{6})) + (\cos(\frac{n\pi}{6}) - i\sin(\frac{n\pi}{6}))\right)$$

$$= 2^n \left(2\cos(\frac{n\pi}{6})\right) = 2^{n+1}\cos(\frac{n\pi}{6}), \text{ which is real.}$$

When $n = 12$, we have

$$(\sqrt{3} + i)^{12} + (\sqrt{3} - i)^{12} = 2^{13}\cos(2\pi) = 2^{13}.$$

[Alternative Solution] We use the result: $z = \bar{z} \iff z$ is real.

Let $z = (\sqrt{3} + i)^n + (\sqrt{3} - i)^n$. Consider the conjugation of z:

$$\bar{z} = \overline{(\sqrt{3} + i)^n + (\sqrt{3} - i)^n} = \overline{(\sqrt{3} + i)^n} + \overline{(\sqrt{3} - i)^n}$$

$$= \left(\overline{\sqrt{3} + i}\right)^n + \left(\overline{\sqrt{3} - i}\right)^n = (\sqrt{3} - i)^n + (\sqrt{3} - i)^n = z$$

Therefore, z is a real number.

Note that the conjugation of $w = (\sqrt{3} + i)^n$ is $\bar{w} = (\sqrt{3} - i)^n$. Thus we have

$$z = w + \bar{w} = 2\text{Re}(w) = 2\text{Re}(\sqrt{3} + i)^n = 2\text{Re}(2\text{cis}\frac{\pi}{6})^n = 2\text{Re}(2^n\text{cis}\frac{n\pi}{6}) = 2^{n+1}\cos\frac{n\pi}{6}.$$

When $n = 12$, $z = 2^{13}\cos\frac{12\pi}{6} = 2^{13}$.

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11. Find all four distinct fourth roots of $-16i$.

[Solution] Note that $-16i = 2^4\text{cis}(\frac{-\pi}{2})$.

Thus the four distinct fourth roots of $-16i$ are

$$z_k = 2\text{cis}(\frac{\frac{-\pi}{2} + 2k\pi}{4}), k = 0, 1, 2, 3;$$

which are

$$z_0 = 2\text{cis}(\frac{-\pi}{8}), \quad z_1 = 2\text{cis}(\frac{3\pi}{8}), \quad z_2 = 2\text{cis}(\frac{7\pi}{8}), \quad z_3 = 2\text{cis}(\frac{11\pi}{8}) = 2\text{cis}(\frac{-5\pi}{8})$$

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12. Solve the following equations.

(a) $z^4 + 4z^2 + 16 = 0$

(b) $z^4 + 1 = 0$

(c) $z^3 + z^2 + z + 1 = 0$

[Solution] Note the these three equations involves real-coefficient polynomials. Thus, if w is a complex root of the polynomial equation, its conjugate $\bar{w}$ will also be a root of the equation.

(a)

$$z^4 + 4z^2 + 16 = 0 \Longleftrightarrow z^2 = \frac{-4 \pm \sqrt{4^2 - 4(16)}}{2} = -2 \pm 2\sqrt{3}i = 4e^{(\pm\frac{2\pi}{3})}i$$

Thus, we have

$$z = 2e^{(\pm\frac{2\pi}{3} + 2k\pi)i/2} = 2e^{\frac{(3k\pm1)\pi}{3}i}, k = 0, 1.$$

The roots are

$$z_0 = 2e^{\frac{\pi}{3}} = 1 + \sqrt{3}i, z_1 = 2e^{\frac{4\pi}{3}} = 2e^{\frac{-2\pi}{3}} = -1 - \sqrt{3}i,$$
$$z_0' = 2e^{\frac{-\pi}{3}} = 1 - \sqrt{3}i, \text{ and } z_1' = 2e^{\frac{2\pi}{3}} = -1 + \sqrt{3}i.$$

Remark: Note that $z^4 + 4z^2 + 16$ is a real-coefficient polynomial of degree 4. Thus a complex root and its conjugate will be roots. Here we have z_0 and z_0' are conjugate pair, while z_1 and z_1' are another conjugate pair.

(b)

$$z^4 + 1 = 0 \Longleftrightarrow z^4 = -1 = e^{\pi i}$$

Thus, we have

$$z = e^{(\frac{2k\pi+\pi}{4}i)} = e^{\frac{(2k+1)\pi}{4}i}, k = 0, 1, 2, 3;$$

which are

$$z_0 = e^{\frac{\pi}{4}i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, z_1 = e^{\frac{3\pi}{4}i} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$$
$$z_2 = e^{\frac{5\pi}{4}i} = e^{\frac{-3\pi}{4}i} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \text{ or } z_3 = e^{\frac{7\pi}{4}i} = e^{\frac{-\pi}{4}i} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$

Remark: Which roots are conjugate pairs?

(c)

$$z^3 + z^2 + z + 1 = 0 \Longleftrightarrow z^2(z+1) + (z+1) = 0 \Longleftrightarrow (z^2+1)(z+1) = 0$$
$$\Longleftrightarrow (z+1) = 0 \text{ or } (z^2+1) = 0$$
$$\Longleftrightarrow z = -1 \text{ or } z^2 = -1 = e^{\pi i}$$
$$\Longleftrightarrow z = -1 \text{ or } z = e^{\frac{\pi}{2}i} = i \text{ or } z = e^{\frac{3\pi}{2}i} = -i$$

Remark: Which roots are conjugate pairs?

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13. Solve the equation $\left(\frac{z-4i}{2i}\right)^3 = i$ and represent the roots of the equation in an Argand diagram.

[Solution]

$$\left(\frac{z-4i}{2i}\right)^3 = i$$

$$\Longleftrightarrow (z-4i)^3 = i(2i)^3 = 8 = 8e^{0i}$$

Thus we have

$$z - 4i = 2e^{i(2k\pi)/3}, k = 0, 1, 2;$$

which gives

$$z = 4i + 2e^{i(2k\pi)/3}, k = 0, 1, 2; i.e., z = 4i + 2 \text{ or } z = 4i + (-1 + i\sqrt{3}) \text{ or } z = 4i + (-1 - i\sqrt{3}).$$

Roots of the given equation are:

$$z = 2 + 4i, \text{ or } z = -1 + i(4 + \sqrt{3}) \text{ or }, z = -1 + i(4 - \sqrt{3}).$$

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14. If α is a complex 5th root of unity with the smallest positive principal argument, determine the value of

$$(1 + \alpha^4)(1 + \alpha^3)(1 + \alpha^2)(1 + \alpha).$$

[Solution] The five distinct 5th root of unity are given by

$$z_k = \text{ cis } \frac{2k\pi}{5} = (\text{ cis } \frac{2\pi}{5})^k, k = 0, 1, 2, 3, 4.$$

Thus, we have $\alpha = \text{ cis } (\frac{2\pi}{5})$. Each α^k is a 5th root of unity, and it satisfies $z^5 - 1 = 0$.
Hence, we factorise $z^5 - 1$ as

$$z^5 - 1 = (z - 1)(z - \alpha)(z - \alpha^2)(z - \alpha^3)(z - \alpha^4).$$

Substituting $z = -1$, we obtain

$$-2 = -2(-1 - \alpha)(-1 - \alpha^2)(-1 - \alpha^3)(-1 - \alpha^4) = -2(1 + \alpha)(1 + \alpha^2)(1 + \alpha^3)(1 + \alpha^4),$$

which gives

$$(1 + \alpha^4)(1 + \alpha^3)(1 + \alpha^2)(1 + \alpha) = 1.$$

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15. Suppose $\sin\dfrac{\theta}{2} \neq 0$. Prove that

$$\frac{1}{2} + \sum_{k=1}^{n} \cos k\theta = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}.$$

(Hint: Let $z = \cos\theta + i\sin\theta$. Geometric sum: $\displaystyle\sum_{k=1}^{n} z^k = z + z^2 + \cdots + z^n = \frac{z(z^n-1)}{z-1}$.)

[Proof.] This problem is OPTIONAL.

We shall prove that

$$\sum_{k=1}^{n} \cos k\theta = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}} - \frac{1}{2}.$$

Note that $\cos k\theta$ is the real part of the complex number $\cos k\theta + i\sin k\theta$, which in turn is $(\cos\theta + i\sin\theta)^k = e^{ik\theta}$, by DeMoivre's Theorem. Let $z = \cos\theta + i\sin\theta = e^{i\theta}$, we have

$$\sum_{k=1}^{n} (\cos k\theta + i\sin k\theta) = \sum_{k=1}^{n} (\cos\theta + i\sin\theta)^k = \sum_{k=1}^{n} z^k = \frac{z(1-z^n)}{1-z}$$

$$= \frac{e^{i\theta}\left(1-e^{i(n\theta)}\right)}{1-e^{i\theta}} = \frac{e^{i\theta}\left(1-e^{i(n\theta)}\right)}{1-e^{i\theta}} \quad \left(\frac{e^{-i\theta/2}}{e^{-i\theta/2}}\right) = \frac{e^{i\theta/2}\left(1-e^{i(n\theta)}\right)}{\left(e^{-i\theta/2}-e^{i\theta/2}\right)} = \frac{e^{i\theta/2}-e^{i(n+1/2)\theta}}{e^{-i\theta/2}-e^{i\theta/2}}$$

$$= \frac{\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right) - \left(\cos\left[\left(n+\frac{1}{2}\right)\theta\right] + i\sin\left[\left(n+\frac{1}{2}\right)\theta\right]\right)}{-2i\sin\frac{\theta}{2}} \quad \left(\frac{i}{i}\right)$$

$$= \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right] - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} + i\frac{\cos\frac{\theta}{2} - \cos\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}.$$

Comparing the real parts of both sides, we have

$$\sum_{k=1}^{n} \cos k\theta = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right] - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}} - \frac{1}{2}.$$

Thus

$$\frac{1}{2} + \sum_{k=1}^{n} \cos k\theta = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}.$$

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16. (a) Use De Moivre's Theorem and binomial expansion to show that

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

(b) Express $\sin 5\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

(c) Hence, obtain an expression for $\tan 5\theta$ in terms of powers of $\tan\theta$.

(Answer: (b) $5c^4s - 10c^2s^3 + s^5$ (c) $\dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$)

[Solution] Let $z = \cos\theta + \text{ß}\sin\theta$. By De Moivre's Theorem, we have $z^5 = \cos 5\theta + i\sin 5\theta$. Thus, $\cos 5\theta = Re(z)$, and $\sin 5\theta = Im(z)$.

Using Binomial Expansion, we have

$$z^5 = \cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= c^5 + i5c^4s - 10c^3s^2 - i10c^2s^3 + 5cs^4 + is^5$$

(a) Comparing the real parts, we have

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

(b) Comparing the imaginary parts, we have

$$\sin 5\theta = \sin^5\theta - 10\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta.$$

(c) Dividing $\sin 5\theta$ by $\cos 5\theta$, we have

$$\tan 5\theta = \frac{\sin^5\theta - 10\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

$$= \frac{\frac{\sin^5\theta}{\cos^5\theta} - \frac{10\cos^2\theta\sin^3\theta}{\cos^5\theta} + \frac{5\cos^4\theta\sin\theta}{\cos^5\theta}}{\frac{\cos^5\theta}{\cos^5\theta} - \frac{10\cos^3\theta\sin^2\theta}{\cos^5\theta} + \frac{5\cos\theta\sin^4\theta}{\cos^5\theta}}$$

$$= \frac{\tan^5\theta - 10\tan^2\theta + 5\tan^4\theta}{1 - 10\tan^3 + 5\tan\theta}$$

17. (a) Let $z = \cos\theta + i\sin\theta$. Show that

(i) $\dfrac{1}{z} = \cos\theta - i\sin\theta$.

(ii) $z^k + \dfrac{1}{z^k} = 2\cos k\theta$ and $z^k - \dfrac{1}{z^k} = 2i\sin k\theta$, for $k \in \mathbb{Z}^+$.

(b) Use the results in part (a) to prove that

$$\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right).$$

Hence find the integral $\displaystyle\int 8\cos^4\theta \; d\theta$.

[Solution]

(a) Let $z = \cos\theta + i\sin\theta$.

(i) $\dfrac{1}{z} = \dfrac{1}{\cos\theta + i\sin\theta} = \dfrac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$.

(ii) By De Moivre's Theorem, we have

$$z^k = \cos(k\theta) + i\sin(k\theta),$$

and hence

$$\frac{1}{z^k} = \frac{1}{\cos(k\theta) + i\sin(k\theta)} = \cos(k\theta) - i\sin(k\theta).$$

Thus, we have $z^k + \dfrac{1}{z^k} = 2\cos k\theta$ and $z^k - \dfrac{1}{z^k} = 2i\sin k\theta$, for $k \in \mathbb{Z}^+$.

(b) Use the results in part (a), we have

$$\begin{aligned}
\cos^4\theta = \left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^4 &= \frac{1}{2^4}\left(z^4 + 4(z^3)\frac{1}{z} + 6z^2(\frac{1}{z})^2 + 4z(\frac{1}{z})^3 + (\frac{1}{z})^4\right) \\
&= \frac{1}{2^4}\left(z^4 + 4z^2 + 6 + 4(\frac{1}{z^2}) + \frac{1}{z^4}\right) \\
&= \frac{1}{2^4}\left((z^4 + \frac{1}{z^4}) + 4(z^2 + \frac{1}{z^2}) + 6\right) \\
&= \frac{1}{2^4}\left(2\cos 4\theta + 4(2\cos 2\theta) + 6\right) \\
&= \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right).
\end{aligned}$$

Hence, we have

$$\int 8\cos^4\theta \; d\theta = \int \cos 4\theta + 4\cos 2\theta + 3 \; d\theta = \frac{\sin 4\theta}{4} + 2\sin 2\theta + 3\theta + C.$$

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