Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics 1** **Tutorial 1**

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1. Evaluate the expression and write your answer in the form $a + bi$, $a, b \in \mathbb{R}$.
   (a) $i^{179} + i^{2013}$
   (b) $\dfrac{1+2i}{3-4i}$

2. For each of the following, represent the complex number on the Argand diagram. Find the modulus and the principal argument of the complex number. Hence express the complex number in its polar representation.
   (a) $1 + \sqrt{3}i$
   (b) $-1 + \sqrt{3}i$
   (c) $1 - \sqrt{3}i$
   (d) $-1 - \sqrt{3}i$

3. Find the complex conjugate of each of the following complex numbers.
   (a) $2i$
   (b) 2
   (c) $1 + 3i$
   (d) $-3 - 4i$
   (e) a complex number with modulus 2 and argument $\theta = \frac{\pi}{3}$.

4. Sketch the regions defined by
   (a) $\mathrm{Re}(z) \geq 0$
   (b) $\mathrm{Im}(z) < 2$
   (c) $|z| \geq 2$
   (d) $\dfrac{\pi}{6} \leq \arg(z) \leq \dfrac{\pi}{3}$
   (e) $|z - i| = |z - 1|$
   (f) $|z - (1 + i)| \leq 2$

5. Solve the following equation for the real numbers, x and y.
$$\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1 + i$$

6. Suppose a complex number $z = x + iy$ satisfies
$$|z - 1| = \frac{1}{2}\,|z - i|\,.$$
   Show that x and y satisfy the following equation
$$\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{8}{9},$$
   which represents a circle of radius $\dfrac{\sqrt{8}}{3}$ centred at $\left(\dfrac{4}{3}, \dfrac{-1}{3}\right)$.

7. Express each of the following complex numbers in the form $re^{i\theta}$, with $r \geq 0$, and $-\pi < \theta \leq \pi$.

   (a) $\left(1+\sqrt{-3}\right)^2$ (b) $\dfrac{1+i}{1-i}$

8. Let $z = a + ib$, where a and b are some real numbers.

   Show that

   (a) $z + \bar{z} = 2\mathrm{Re}(z)$.
   (b) $z = \bar{z} \iff z$ is a real number.
   (c) If z is a root of $ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are real constants, then $\bar{z}$ is also a root of $ax^3 + bx^2 + cx + d = 0$.

   Remark *More generally, we have*

   Suppose that $p(x) = a_0 + a_1 x + ... + a_n x^n$ is a polynomial in x with real coefficients a_k's. If a complex number z is a solution of $p(x) = 0$, then the conjugate $\overline{z}$ of z is also a solution of $p(x) = 0$.

9. Express each of following complex numbers in the form (i) $\cos\alpha + i\sin\alpha$ and (ii) $x + iy$.

   (a) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^7$
   (b) $\dfrac{1}{\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^2}$
   (c) $\dfrac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^4}$

10. Without using any series expansions, prove that

$$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n \text{ is real.}$$

   Find the value of this expression when $n = 12$.

11. Find all four distinct fourth roots of $-16i$.

12. Solve the following equations.

   (a) $z^4 + 4z^2 + 16 = 0$
   (b) $z^4 + 1 = 0$
   (c) $z^3 + z^2 + z + 1 = 0$

13. Solve the equation $\left(\dfrac{z-4i}{2i}\right)^3 = i$ and represent the roots of the equation in an Argand diagram.

   (a) Find the cube roots of -1 and show that they can be denoted by -1, λ , $-\lambda^2$.
   (b) Consider the factorization

$$(z^3+1) = (z+1)(z-\lambda)(z+\lambda^2).$$

   By comparing the coefficients of z^2, prove that $\lambda^2 - \lambda + 1 = 0$. What is the value of $(2-\lambda)(2+\lambda^2)$? (Answer: 3)

14. If α is a complex 5th root of unity with the smallest positive principal argument, determine the value of

$$(1+\alpha^4)(1+\alpha^3)(1+\alpha^2)(1+\alpha).$$

15. Suppose $\sin\dfrac{\theta}{2} \neq 0$. Prove that

$$\frac{1}{2}+\sum_{k=1}^{n}\cos k\theta = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}.$$

(Hint: Let $z=\cos\theta+i\sin\theta=e^{i\theta}$. Geometric sum: $\displaystyle\sum_{k=1}^{n} z^k = z+z^2+\cdots+z^n = \frac{z(z^n-1)}{z-1}$.)

16. (a) Use De Moivre's Theorem and binomial expansion to show that

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

(b) Express $\sin 5\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

(c) Hence, obtain an expression for $\tan 5\theta$ in terms of powers of $\tan\theta$.

(Answer: (b) $5c^4s-10c^2s^3+s^5$ (c) $\dfrac{5t-10t^3+t^5}{1-10t^2+5t^4}$)

17. (a) Let $z=\cos\theta+i\sin\theta$. Show that

(i) $\dfrac{1}{z}=\cos\theta-i\sin\theta$, and hence $\cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$ and $\sin\theta=\dfrac{1}{2i}\left(z-\dfrac{1}{z}\right)$.

(ii) $z^k+\dfrac{1}{z^k}=2\cos k\theta$ and $z^k-\dfrac{1}{z^k}=2i\sin k\theta$, for $k\in\mathbb{Z}^+$.

(b) Use the results in part (a) to prove that

$$\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right).$$

Hence find the integral $\displaystyle\int 8\cos^4\theta\, d\theta$.

Challenging Problem (Optional. Will not be discussed in the tutorial session.)

1. Consider a regular n-sided polygon circumscribed by the unit circle. Prove that the product of the length of line segments formed by joining one vertex of the regular polygon to the rest of the vertices equals to n: (Hint: Consider the n^{th} roots of 1).