Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 12**

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1. Find the values of p for which the integral converges

(a) $\displaystyle\int_1^2 \frac{1}{x(\ln x)^p}\ dx$

(b) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^p}\ dx$

Solution By letting $u = \ln x$, we have

$$\begin{aligned}
\int \frac{1}{x(\ln x)^p} dx &= \int \frac{1}{u^p}\frac{du}{dx} dx \\
&= \int \frac{1}{u^p} du \\
&= \begin{cases} \frac{u^{-p+1}}{1-p} & \text{if } p \neq 1 \\ \ln|u| & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \frac{(\ln|x|)^{1-p}}{1-p} & \text{if } p \neq 1 \\ \ln|\ln x| & \text{if } p = 1 \end{cases} .
\end{aligned}$$

(a)

$$\begin{aligned}
\int_1^2 \frac{1}{x(\ln x)^p}\ dx &= \lim_{t\to1^{-1}} \int_t^2 \frac{1}{x(\ln x)^p}\ dx \\
&= \lim_{t\to1^{-1}} \begin{cases} \frac{(\ln 2)^{1-p}}{1-p} - \frac{(\ln t)^{1-p}}{1-p} & \text{if } p \neq 1 \\ \ln(\ln 2) - \ln|\ln t| & \text{if } p = 1 \end{cases} \\
&= \lim_{t\to1^{-1}} \begin{cases} \frac{(\ln 2)^{1-p}}{1-p} - \frac{(\ln t)^{1-p}}{1-p} & \text{if } p < 1 \\ \frac{(\ln 2)^{1-p}}{1-p} - \frac{(\ln t)^{1-p}}{1-p} & \text{if } p > 1 \\ \ln(\ln 2) - \ln|\ln t| & \text{if } p = 1 \end{cases} \\
&= \lim_{t\to1^{-1}} \begin{cases} \frac{(\ln 2)^{1-p}}{1-p} & \text{if } p < 1 \\ \text{diverge} & \text{if } p > 1 \\ \text{diverge} & \text{if } p = 1 \end{cases}
\end{aligned}$$

The integral converges when $p < 1$ and diverges when $p \geq 1$.

(b)

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^p}\,dx \\
&= \lim_{t\to\infty} \begin{cases} \frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} & \text{if } p<1 \\ \frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} & \text{if } p>1 \\ \ln|\ln t| - \ln(\ln 2) & \text{if } p=1 \end{cases} \\
&= \lim_{t\to\infty} \begin{cases} \text{diverge} & \text{if } p<1 \\ -\frac{(\ln 2)^{1-p}}{1-p} & \text{if } p>1 \\ \text{diverge} & \text{if } p=1 \end{cases}
\end{aligned}$$

The integral converges when $p>1$ and diverges when $p\le 1$.

2. Estimate each of the following definite integrals using the Trapezoidal Rule with $n=4$.

(a) $\int_1^2 x\ dx$ (b) $\int_1^3 (2x-1)\ dx$

Solution

Trapezoidal Rule with $n=4$.

$$\int_a^b f(x)\ dx \approx T_4 = \frac{\Delta x}{2}(y_0+2y_1+2y_2+2y_3+y_4).$$

(a) $[a,b]=[1,2]$, then $x_0=1, x_1=5/4,\ x_2=6/4,\ x_3=7/4,\ x_4=2$ and $\Delta x=\frac{1}{4}$.

n	x_n	y_n
0	1	1
1	1.25	1.25
2	1.5	1.5
3	1.75	1.75
4	2	2

Then

$$\begin{aligned}
\int_1^2 x\,dx \approx T_4 &= \frac{\Delta x}{2}(y_0+2y_1+2y_2+2y_3+y_4) \\
&= \frac{1/4}{2}(1+2(1.25)+2(1.5)+2(1.75)+2) \\
&= \frac{1}{8}(1+2.5+3+3.5+2) = \frac{12}{8} = \frac{3}{2}.
\end{aligned}$$

(b) $[a,b]=[1,3]$, then $x_0=1, x_1=1.5,\ x_2=2,\ x_3=2.5,\ x_4=3$ and $\Delta x=\frac{1}{2}$.

n	x_n	y_n
0	1	1
1	1.5	2
2	2	3
3	2.5	4
4	3	5

Then

$$\begin{aligned}\int_1^3 (2x-1)\,dx \approx T_4 &= \frac{\Delta x}{2}\left(y_0+2y_1+2y_2+2y_3+y_4\right)\\ &= \frac{1/2}{2}\left(1+2\,(2)+2\,(3)+2\,(4)+5\right)\\ &= \frac{1}{4}\left(1+4+6+8+5\right) = \frac{24}{4} = 6.\end{aligned}$$

3. Estimate each of the following definite integrals using Simpson's Rule with $n=4$.

(a) $\int_{-1}^{1}\left(x^2+1\right)\ dx$ (b) $\int_{-2}^{0}\left(x^2-1\right)\ dx$

Solution

Simpson's Rule with $n=4$.

$$S_4 = \frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+y_4\right).$$

(a) $[a,b]=[-1,1]$, then $x_0=-1, x_1=-1/2,\ x_2=0,\ x_3=1/2,\ x_4=1$ and $h=\frac{1}{2}$.

n	x_n	y_n
0	-1	2
1	$-1/2$	1.25
2	0	1
3	$1/2$	1.25
4	1	2

Then

$$\begin{aligned}\int_{-1}^{1}\left(x^2+1\right)dx \approx S_4 &= \frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+y_4\right)\\ &= \frac{1/2}{3}\left(2+4\,(1.25)+2\,(1)+4\,(1.25)+2\right)\\ &= \frac{1}{6}\,(16) = \frac{8}{3}.\end{aligned}$$

(b) $[a,b]=[-2,0]$, then $x_0=-2, x_1=-1.5,\ x_2=-1,\ x_3=-0.5,\ x_4=0$ and $\Delta x=\frac{1}{2}$.

n	x_n	y_n
0	-2	3
1	-1.5	1.25
2	-1	0
3	-0.5	-0.75
4	0	-1

$$\begin{aligned}\int_{-2}^{0}\left(x^2-1\right)\ dx \approx S_4 &= \frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+y_4\right)\\ &= \frac{1/2}{3}\left(3+4\,(1.25)+2\,(0)+4\,(-0.75)+(-1)\right)\\ &= \frac{2}{3}.\end{aligned}$$

4. Prove that the volume of the cone with height h and radius r is $\frac{1}{3}\pi r^2 h$.

Solution

Consider $\frac{y}{h} + \frac{x}{r} = 1$, *i.e.*, $y = h\left(1 - \frac{x}{r}\right)$.

Then the solid formed by revolving the area bounded the line $y = h\left(1 - \frac{x}{r}\right)$ and the x and y axes is a cone height h and radius r.

Cylindrical Shell Method

Height of a typical shell $= h\left(1 - \frac{x}{r}\right)$,

Circumference of a typical shell $= 2\pi x$

Volume of a typical shell $= 2\pi x h\left(1 - \frac{x}{r}\right)\delta x$.

Therefore

$$\begin{aligned} V &= \int_0^r 2\pi x h\left(1 - \frac{x}{r}\right) dx \\ &= 2\pi h \int_0^r \left(x - \frac{x^2}{r}\right) dx \\ &= 2\pi h \left[\frac{r^2}{2} - \frac{r^3}{3r}\right] \\ &= \frac{1}{3}\pi r^2 h \end{aligned}$$

5. (a) The equation of a circle with center at the origin and radius r is described by the equation $x^2 + y^2 = r^2$. Use integration to prove that the area of the circle is πr^2.

   (b) When the region bounded by the x-axis and the curve $y = \sqrt{r^2 - x^2}$ for $-r \le x \le r$ is rotated about the x-axis, a sphere with radius r is obtained. Use integration to prove that the volume of the sphere is given by $\frac{4}{3}\pi r^3$.

Solution

(a) Equation of circle : $x^2 + y^2 = r^2$,

The circle is enclosed by $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$.

The area of a typical strip: $A(x) = \left[\sqrt{r^2 - x^2} - \left(-\sqrt{r^2 - x^2}\right)\right]\delta x = 2\sqrt{r^2 - x^2}\delta x$.

Therefore the area of the circle is

$$A = \int_{-r}^{r} 2\sqrt{r^2 - x^2}dx.$$

To evaluate $\int 2\sqrt{r^2 - x^2}dx$, let $x = r\sin\theta$. Then $\frac{dx}{d\theta} = r\cos\theta$. So $1 = r\cos\theta\frac{d\theta}{dx}$

$$\begin{aligned}
\int 2\sqrt{r^2 - x^2}dx &= \int 2\sqrt{r^2 - r^2\sin^2\theta}dx \\
&= \int 2r\sqrt{1 - \sin^2\theta}r\cos\theta\frac{d\theta}{dx}dx \\
&= \int 2r^2\cos^2\theta d\theta \\
&= 2r^2\int \frac{\cos 2\theta + 1}{2}d\theta \\
&= r^2\left[\frac{\sin 2\theta}{2} + \theta\right]
\end{aligned}$$

When $x = r = r\sin\theta$, $\sin\theta = 1$ and thus $\theta = \frac{\pi}{2}$. When $x = -r$, $r\sin\theta = -r$, thus $\theta = -\frac{\pi}{2}$.

Hence,

$$A = \int_{-r}^{r} 2\sqrt{r^2 - x^2}dx = r^2\left[\frac{\sin 2\theta}{2} + \theta\right]_{-\pi/2}^{\pi/2} = \pi r^2.$$

(b) A typical cross section is a <u>disc</u> of radius $\sqrt{r^2 - x^2}$.

Therefore the volume of a typical cross section is $\pi\left(\sqrt{r^2 - x^2}\right)^2\delta x$.

Hence volume of the solid is

$$\begin{aligned}
V &= \int_{-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2 dx \\
&= \pi\int_{-r}^{r} \left(r^2 - x^2\right) dx \\
&= \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} \\
&= \frac{4}{3}\pi r^3.
\end{aligned}$$

6. Use integration by substitution to prove the following.

(a) $\displaystyle\int \tan x dx = \ln|\sec x| + C$

(b) $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + C$

(c) $\displaystyle\int \sin^3 x \cos^8 x dx = -\frac{\cos^9 x}{9} + \frac{\cos^{11}}{11} + C$

Solution

(a) $\displaystyle\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$

Let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$. Therefore

$$\begin{aligned}\int \frac{\sin x}{\cos x} dx &= \int \frac{-\frac{du}{dx}}{\cos x} dx \\ &= \int -\frac{1}{u} du \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + c \\ &= \ln|\sec x| + c.\end{aligned}$$

(b)

$$\begin{aligned}\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx.\end{aligned}$$

Let $u = \sec x + \tan x$. The $\frac{du}{dx} = \sec x \tan x + \sec^2 x$. Therefore

$$\begin{aligned}\int \sec x dx &= \int \frac{du/dx}{\sec x + \tan x} dx \\ &= \int \frac{1}{u} du \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

(c)

$$\begin{aligned}\int \sin^3 x \cos^8 x dx &= \int \sin x \sin^2 x \cos^8 x dx \\ &= \int \sin x \left(1 - \cos^2 x\right) \cos^8 x dx.\end{aligned}$$

Let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$. Then

$$\begin{aligned}
\int \sin^3 x \cos^8 x dx &= \int \sin x \left(1 - \cos^2 x\right) \cos^8 x dx \\
&= \int -\frac{du}{dx} \left(1 - u^2\right) u^8 dx \\
&= -\int \left(u^8 - u^{10}\right) du \\
&= -\frac{u^9}{8} + \frac{u^{11}}{11} + C \\
&= -\frac{\cos^9 x}{9} + \frac{\cos^{11} x}{11} + C.
\end{aligned}$$