Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 11**

Topics: Reduction Formulae, Improper Integrals, Areas, Volumes.

1. Let $I_n = \int \cos^n x\, dx$ for $n = 0, 1, 2, 3, \cdots$.

   (a) Prove the reduction formula

   $$I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \text{ for } n \geq 2.$$

   (b) Use part(a) to evaluate

   (i) $\int \cos^3 x\ dx$.

   (ii) $\int_0^{\pi/2} \cos^4 x\ dx$.

2. Evaluate each of the following improper integrals.

   (a) $\int_0^1 x \ln x\ dx$

   (b) $\int_0^1 \frac{4r}{\sqrt{1-r^4}} dr$

   (c) $\int_1^\infty \frac{1}{x^3}\ dx$

   (d) $\int_5^\infty \frac{1}{\sqrt{x-1}}\ dx$

   (e) $\int_2^\infty \frac{1}{x(\ln x)^2}\ dx$

3. Sketch each of the region enclosed by the given lines and curves. Find the area of the enclosed region.

   (a) $y = 2x - x^2$ and $y = -3$

   (b) $y = x^2 - 2x$ and $y = x$

   (c) $x = y^2$ and $x = y + 2$

   (d) $x = y^3 - y^2$ and $x = 2y$

4. (a) Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

   (b) Sketch the region bounded by the given curves and find the area of the region:

   $$y = \sin x, y = e^x, x = 0, x = \pi/2$$

5. The base of a solid S is circular disk with radius r. Parallel cross-sections perpendicular to the base are squares. Show that the volume of the solid S is $\frac{16}{3}r^3$.

   (Note that the equation of a circle with radius r and centered at $(0, 0)$ is $x^2 + y^2 = r^2$.)

6. (a) Find the volume of the solid obtained by revolving the region bounded by the curves $x = y - y^2$ and $x = 0$ about the y-axis.
   (b) Find the volume of the solid generated by revolving the regions bounded by the curve $y = x^3$ and lines $y = 0$ and $x = 2$ about the x-axis.
   (c) Find the volume of the solid obtained by revolving the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 2$.

7. Find the volume of the solid generated by revolving the regions bounded by the lines and curves about the y-axis.
   (a) $x = y^{3/2}$, $x = 0$, $y = 2$
   (b) $x = \sqrt{2 \sin 2y}$, $0 \leq y \leq \pi/2$, $x = 0$

8. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves $x + y = 3$ and $x = 4 - (y - 1)^2$ about the x-axis.

9. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$ and $x = 1$.
   (a) Find the area of the region.
   (b) Find the volume of the solid formed by revolving this region about the y-axis.

Answers

1. (b) (i) $\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$.
   (ii) $\frac{3\pi}{16}$

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   (a) $-\frac{1}{4}$.
   (b) π.
   (c) $\frac{1}{2}$.
   (d) diverges
   (e) $\frac{1}{\ln 2}$.

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2. Sketch each of the region enclosed by the given lines and curves. Find the area of the enclosed region.
   (a) $\frac{32}{3}$
   (b) $\frac{9}{2}$
   (c) $\frac{9}{2}$
   (d) $\frac{37}{12}$

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3. (a) $\frac{1}{12}$.
   (b) $e^{\frac{\pi}{2}} - 2$.

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6. (a) $\frac{\pi}{30}$
   (b) $\frac{128\pi}{7}$

(c) $\frac{8}{15}\pi$

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7. (a) 4π
   (b) 2π

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8. $\frac{27\pi}{2}$

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9. (a) $\frac{\pi}{4} - \frac{1}{2}\ln 2$
   (b) $\pi(\frac{\pi}{2} - 1)$.