Nanyang Technological University
SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 10**

Topics Fundamental Theorem of Calculus, Basic integration techniques: Substitution, by-parts, partial fractions.

PRACTICE MAKES PERFECT!! To master techniques in integration, firstly, we have to familiarize with the antiderivatives of basic functions. This is important in the technique by substitution and integration-by-parts. Secondly, understand various basic techniques in integration. Through practice and observation, you will be able to summarize when to use which techniques. Sometimes, we have to try out different techniques.

1. Find each of the following derivatives.

   (a) $\dfrac{d}{dx}\left(\displaystyle\int_1^x (2+t^4)^5\,dt\right)$

   (b) $\dfrac{d}{dx}\left(\displaystyle\int_{1/x^2}^0 \sin^3 t\,dt\right)$

   (c) $\dfrac{d}{dx}\left(\displaystyle\int_{\cos x}^{5x} \cos(u^2)\,du\right)$

   [Solution]

   (a) Since the function $f(t) = (2+t^4)^5$ is continuous, by the Fundamental Theorem of Calculus, we have
   $$\frac{d}{dx}\left(\int_1^x (2+t^4)^5\ dt\right) = (2+x^4)^5.$$

   (b)
   $$\frac{d}{dx}\left(\int_{1/x^2}^0 \sin^3 t\ dt\right) = \frac{d}{dx}\left(-\int_0^{1/x^2} \sin^3 t\ dt\right) = -\frac{d}{dx}\left(\int_0^{1/x^2} \sin^3 t\ dt\right)$$
   $$= -\frac{d}{du}\left(\int_0^u \sin^3 t\ dt\right)\cdot\frac{du}{dx}, \text{where } u = 1/x^2$$
   $$= -[\sin^3(u)](-2/x^3) = \frac{2}{x^3}\sin^3\left(\frac{1}{x^2}\right)$$

   (c) Note that
   $$\int_{\cos x}^{5x} \cos(u^2)du = \int_0^{5x} \cos(u^2)du - \int_0^{\cos x} \cos(u^2)du$$
   Thus,
   $$\frac{d}{dx}\left(\int_{\cos x}^{5x} \cos(u^2)du\right) = \frac{d}{dx}\left(\int_0^{5x} \cos(u^2)du - \int_0^{\cos x} \cos(u^2)du\right)$$
   $$= \frac{d}{dv}\left(\int_0^v \cos(u^2)du\right)\cdot\frac{dv}{dx} - \frac{d}{dw}\left(\int_0^w \cos(u^2)du\right)\cdot\frac{dw}{dx}, \text{where } v = 5x, w = \cos x$$
   $$= (\cos v^2)5 - (\cos w^2)(-\sin x) = 5\cos 25x^2 + (\sin x)\cos(\cos^2 x)$$

2. Find a function $f(x)$ and a value for the constant c such that

$$\int_c^x tf(t)\ dt = \sin x - x\cos x - \frac{1}{2}x^2, \text{ for all real } x.$$

(Answer: $f(x) = \sin x - 1, c = 0$.)

[Solution]

Substituting the given equation with $x = c$, we obtain

$$0 = \sin c - c\cos c - \frac{c^2}{2}.$$

By inspection, we may choose $c = 0$.

Differentiating the above equation with respect to x, we obtain

$$xf(x) = \cos x - (\cos x - x\sin x) - x = x(\sin x - 1).$$

Thus, we may choose $f(x) = \sin x - 1$, for all real x.

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3. Evaluate the following definite integrals.

(a) $\displaystyle\int_1^3 \left(5 - \frac{x}{2} + \frac{3}{x^2} - \frac{1}{x}\right)\ dx = \left[5x - \frac{x^2}{4} + \frac{(-1)3}{x} - \ln|x|\right]_1^3 = 10 - \frac{8}{4} - 3(\frac{1}{3} - 1) - \ln 3 = 10 - \ln 3$

(b) $\displaystyle\int_{-1}^1 (x^2 - 2x + 3)\ dx = \left[\frac{x^3}{3} - x^2 + 3x\right]_{-1}^1 = \frac{2}{3} + 6 = \frac{20}{3}$

(c) $\displaystyle\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\ dx = \left[\frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2}\right]_1^9 = \frac{2}{3}(27 - 1) + 2(3 - 1) = \frac{64}{3}$

(d) $\displaystyle\int_0^{\pi/4} \left(1 + \cos x - \underbrace{\tan^2 x}_{\sec^2 x - 1}\right)\ dx = [x + \sin x - (\tan x - x)]_0^{\pi/4} = \frac{\pi}{2} + \frac{1}{\sqrt{2}} - 1$

(e) $\displaystyle\int_0^{\pi/4} (\sin(2x) - \cos(5x))\ dx = \left[\frac{-\cos(2x)}{2} - \frac{\sin(5x)}{5}\right]_0^{\pi/4} = \frac{1}{2} - \frac{\sin(5\pi/4)}{5} = \frac{1}{2} + \frac{1}{5\sqrt{2}}$

(f) $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x} dx = \int_{\pi/2}^{\pi} \frac{2\sin x\cos x}{2\sin x} dx = \int_{\pi/2}^{\pi} \cos x\ dx = [\sin x]_{\pi/2}^{\pi} = -1$

(g) $\displaystyle\int_{-1}^0 (2^u + e^u)\ du = \left[\frac{2^u}{\ln 2} + e^u\right]_{-1}^0 = (\frac{1}{\ln 2} + 1) - (\frac{1}{2\ln 2} + e^{-1}) = \frac{1}{2\ln 2} + 1 - \frac{1}{e}$

(h) $\displaystyle\int_{-\pi}^{\pi/2} f(x)\,dx$ where $f(x) = \begin{cases} e^x & \text{if } -\pi \le x \le 0, \\ \cos x & \text{if } 0 < x \le \pi. \end{cases}$

[Solution]

$$\int_{-\pi}^{\pi/2} f(x)dx = \int_{-\pi}^{0} f(x)\ dx + \int_{0}^{\pi/2} f(x)\ dx = \int_{-\pi}^{0} e^x\ dx + \int_{0}^{\pi/2} \cos x\ dx$$
$$= [e^x]_{-\pi}^{0} + [\sin x]_0^{\pi/2} = e^0 - e^{-\pi} + (\sin(\pi/2) - \sin(0) = 2 - e^{-\pi}$$

(i) $\displaystyle\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\ dx = \left[\sin^{-1} x\right]_0^{0.5} = \frac{\pi}{6}.$

(j) $\displaystyle\int_1^{\sqrt{3}} \frac{1}{1+x^2}\ dx = \left[\tan^{-1} x\right]_1^{\sqrt{3}} = \tan^{-1}(\sqrt{3}) - (\tan^{-1}(1)) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$

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4. Evaluate each of the following integrals by an appropriate substitution.

(a) $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\ dx = \int \underbrace{(\tan^{-1} x)}_{u} \underbrace{\frac{1}{1+x^2}}_{u'}\ dx = \int u\ du = \frac{u^2}{2} + C = \frac{(\tan^{-1} x)^2}{2} + C$

(b) $\displaystyle\int \frac{e^x}{e^x+1}\ dx = \int \frac{1}{\underbrace{e^x+1}_{u}} \underbrace{e^x}_{u'}\ dx = \int \frac{1}{u}\ du = \ln|u| + C = \ln(1+e^x) + C$

(c) $\displaystyle\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\ dx = \int_0^{1/2} \underbrace{\sin^{-1} x}_{u} \underbrace{\frac{1}{\sqrt{1-x^2}}}_{u'}\ dx = \int_0^{\pi/6} u\ du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}$

(d) $\displaystyle\int 7\sqrt{7x-1}\ dx$ (Let $u = 7x - 1$, $u' = 7$.)
$\displaystyle= \int \sqrt{u}\ du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(7x-1)^{3/2} + C$

(e) $\displaystyle\int \frac{4x^3}{(x^4+1)^2}\ dx$ (let $u = x^4 + 1$, $u' = 4x^3$)
$\displaystyle= \int \frac{1}{u^2}\ du = \frac{-1}{u} + C = \frac{-1}{x^4+1} + C$

(f) $\displaystyle\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx =$ (Let $u = (1+\sqrt{x})$, $u' = \frac{1}{2\sqrt{x}}$.)
$\displaystyle= 2\int (1+\sqrt{x})^{1/3}\ \frac{1}{2\sqrt{x}} dx$
$\displaystyle= 2\int u^{1/3}\ du = 2\left(\frac{u^{4/3}}{4/3}\right) + C = \frac{3(1+\sqrt{x})^{4/3}}{2} + C$

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5. Evaluate the following integrals.

(a) $\displaystyle\int x \ln x \; dx$

(b) $\displaystyle\int \sin^{-1} y \; dy$

(c) $\displaystyle\int xe^{-x} \, dx$

(d) $\displaystyle\int x \sin(\pi x) \, dx$

(e) $\displaystyle\int \sin(\ln x) \; dx.$

[Solution]

(a) $\displaystyle\int x \ln x \; dx$

Integration by parts: $\displaystyle\int u(x)v'(x) \; dx = u(x)v(x) - \int u'(x)v(x) \; dx$

$u = \ln x$ and $v' = x$
$u' = \frac{1}{x}$ and $v = \frac{x^2}{2}$

$$\int x \ln x \; dx = \frac{x^2}{2}(\ln x) - \int (\frac{1}{x})\frac{x^2}{2} \; dx = \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \; dx$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

(Exercise: Check your answer: Is $\displaystyle\frac{d}{dx}\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right) = x \ln x$?)

(b) $\displaystyle\int \sin^{-1} y \; dy$

Integration by parts:
$u = \sin^{-1} y$ and $v' = 1$
$u' = \frac{1}{\sqrt{1-y^2}}$ and $v = y$

$$\int \sin^{-1} y \; dy = y \sin^{-1} y - \int \frac{y}{\sqrt{1-y^2}} \; dy = y \sin^{-1} y - \frac{1}{2}\int \frac{1}{\sqrt{1-t}} \; dt \qquad \text{substitution } t = y^2$$

$$= y \sin^{-1} y + \frac{1}{2}\left(\frac{\sqrt{1-t}}{1/2}\right) + C = y \sin^{-1} y + \sqrt{1-y^2} + C$$

(c) $\displaystyle\int xe^{-x} \, dx$

Using integration by parts,
with $u = x$ and $v' = e^{-x}$,
we have $u' = 1$ and $v = -e^{-x}$. Thus, we have

$$\int xe^{-x} \, dx = -xe^{-x} - \int (-e^{-x}) dx = -xe^{-x} - e^{-x} + C$$

(d) $\displaystyle\int x\sin(\pi x)\,dx$

Integration by parts:

$u = x$ and $v' = \sin(\pi x)$

$u' = 1$ and $v = \frac{-\cos(\pi x)}{\pi}$

$$\begin{aligned}\int x\sin(\pi x)\,dx &= \frac{-x\cos(\pi x)}{\pi} - \int \frac{-\cos(\pi x)}{\pi}dx \\ &= \frac{-x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2} + C\end{aligned}$$

(e) $\displaystyle\int \sin(\ln x)\ dx.$

Integration by parts:

$u = \sin(\ln x)$ and $v' = 1$

$u' = \frac{\cos(\ln x)}{x}$ and $v = x$

$$\int \sin(\ln x)\ dx = x\sin(\ln x) - \int x\left(\frac{\cos(\ln x)}{x}\right)\ dx = x\sin(\ln x) - \int \cos(\ln x)\ dx - - - (1)$$

We solve $\displaystyle\int \cos(\ln x)\ dx$ using integration by parts:

$u = \cos(\ln x)$ and $v' = 1$

$u' = \frac{-\sin(\ln x)}{x}$ and $v = x$

$$\int \cos(\ln x)\ dx = x\cos(\ln x) - \int x\left(\frac{-\sin(\ln x)}{x}\right)\ dx = x\cos(\ln x) + \int \sin(\ln x)\ dx - - - (2)$$

Replacing (2) into (1), we have,

$$\int \sin(\ln x)\ dx = x\sin(\ln x) - \left(x\cos(\ln x) + \int \sin(\ln x)\ dx\right) = x\sin(\ln x) - x\cos(\ln x) - \underbrace{\int \sin(\ln x)\ dx}_{\text{shift to left}}.$$

Shifting the term $\int \sin(\ln x)\ dx$ gives

$$2\int \sin(\ln x)\ dx = x\sin(\ln x) - x\cos(\ln x).$$

Hence we have

$$\int \sin(\ln x)\ dx = \frac{1}{2}\left(x\sin(\ln x) - x\cos(\ln x)\right) + C.$$

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6. Evaluate the integrals.

(a) $\displaystyle\int \frac{x-1}{x^2+3x+2} dx$ (Answer: $-2\ln|x+1| + 3\ln|x+2| + C$)

(b) $\displaystyle\int \frac{x+4}{x^2+5x+6}\ dx$

(c) $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2} dx$ (Answer: $\frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C$)

(d) $\displaystyle\int \frac{x^3}{(x+1)^3} dx$ (Answer: $x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C$)

(e) $\displaystyle\int \frac{1}{x^2+16}\ dx$

(f) $\displaystyle\int \frac{1}{x^2+2x+5}\ dx$

(g) $\displaystyle\int \frac{x}{x^2+4x+13} dx$ (Answer: $\ln(x^2+4x+13) - \frac{2}{3}\tan^{-1}(\frac{x+2}{3}) + C$)

[Solution]

(a) $\displaystyle\int \frac{x-1}{x^2+3x+2} dx$ (Answer: $-2\ln|x+1| + 3\ln|x+2| + C$)

[Solution] Note that $x^2+3x+2 = (x+2)(x+1)$. Thus we express $\dfrac{x-1}{x^2+3x+2}$ into partial fractions:

$$\frac{x-1}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{x+2}$$

which gives $x - 1 = A(x+2) + B(x+1)$.
Solving for A and B, we have

$$\frac{x-1}{x^2+3x+2} = \frac{-2}{(x+1)} + \frac{3}{x+2}$$

$$\int \frac{x-1}{x^2+3x+2}\ dx = \int \frac{-2}{(x+1)} + \frac{3}{x+2}\ dx = -2\ln|x+1| + 3\ln|x+2| + C$$

(b) Expressing into partial fractions, we have

$$\frac{x+4}{x^2+5x+6} = \frac{x+4}{(x+3)(x+2)} = \frac{-1}{x+3} + \frac{2}{x+2}.$$

Thus, we have

$$\int \frac{x+4}{x^2+5x+6}\ dx = \int \frac{-1}{x+3} + \frac{2}{x+2}\ dx = -\ln|x+3| + 2\ln|x+2| + C$$

(c) $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2} dx$ (Answer: $\frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C$)

[Solution]

$$\begin{aligned}\int \frac{x^2}{(x-3)(x+2)^2}\ dx &= \int \frac{9/25}{x-3} + \frac{16/25}{x+2} + \frac{-4/5}{(x+2)^2}\ dx \\ &= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C\end{aligned}$$

(d) $\displaystyle\int \frac{x^3}{(x+1)^3}dx$ (Answer: $x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C$)

[Solution] Note that

$$\frac{x^3}{(x+1)^3} = \frac{x^3}{(x^3+3x^2+3x+1)} = 1 - \frac{3x^2+3x+1}{(x^3+3x^2+3x+1)}$$

Partial fractions of $\displaystyle\frac{3x^2+3x+1}{(x^3+3x^2+3x+1)}$:

$$\frac{3x^2+3x+1}{(x^3+3x^2+3x+1)} = \frac{3x^2+3x+1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Solving for A, B and C, we have

$$\frac{x^3}{(x+1)^3} = 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}.$$

Therefore, we have

$$\int \frac{x^3}{(x+1)^3}\,dx = \int 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}\,dx$$
$$= x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C$$

[Alternative solution] We use substitution $y = x+1$:

$$\int \frac{x^3}{(x+1)^3}dx = \int \frac{(y-1)^3}{y^3}dy = \int \frac{(y^3-3y^2+3y-1)}{y^3}dy = \int \left(1 - 3\frac{1}{y} + 3y^{-2} - y^{-3}\right)\,dy$$

$$= y - 3\ln|y| - 3y^{-1} - \frac{y^{-2}}{-2} + C = (x+1) - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C$$

(e) $\displaystyle\int \frac{1}{x^2+16}\,dx$

[Solution] We use the formula: $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$

$$\int \frac{1}{x^2+16}\,dx = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$$

(f) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$

Note that the discriminant of x^2+2x+5 is $b^2-4ac = (2)^2 - 4(1)(5) = -16 < 0$.
Therefore, the quadratic function x^2+2x+5 is irreducible.
Completing square, we have $x^2+2x+5 = (x+1)^2+4$. Thus we have

$\displaystyle\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx = \int \frac{1}{u^2+2^2}\,du$, where $u = x+1$.

$$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.$$

(g) $\displaystyle\int \frac{x}{x^2+4x+13}dx$ (Answer: $\ln(x^2+4x+13) - \frac{2}{3}\tan^{-1}(\frac{x+2}{3}) + C$)

[Solution]

Note that $x^2+4x+13$ is irreducible because its discriminant $b^2-4ac = 16-4(1)(13) < 0$.

By completing square, we have $x^2+4x+13 = (x+2)^2+9$.

Express $\dfrac{x}{x^2+4x+13}$ in the form

$$\frac{A(2x+4)}{x^2+4x+13} + \frac{B}{x^2+4x+13}$$

where the numerator of the first term $\dfrac{(2x+4)}{x^2+4x+13}$ is the derivative of the denominator.

Note that $A = \frac{1}{2}$ and hence $B = -2$.

Thus, we obtain

$$\int \frac{x}{x^2+4x+13}dx$$

$$= \int \frac{\frac{1}{2}(2x+4)}{x^2+4x+13} - \frac{2}{x^2+4x+13}\, dx$$

$$= \frac{1}{2}\ln(x^2+4x+13) - 2\int \frac{1}{(x+2)^2+9}\, dx$$

$$= \frac{1}{2}\ln(x^2+4x+13) - \frac{2}{3}\tan^{-1}(\frac{x+2}{3}) + C$$

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7. Evaluate $\displaystyle\int x^3 \tan^{-1} x\, dx$.

[Solution] We use integration by parts: $u = \tan^{-1} x$ and $v' = x^3$ so that $u' = \dfrac{1}{1+x^2}$ and $v = \dfrac{x^4}{4}$.

$$\int x^3 \tan^{-1} x\, dx$$

$$= \frac{x^4}{4}\ \tan^{-1} x - \int \frac{x^4}{4}\ \frac{1}{1+x^2}\, dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\int \left((x^2-1) + \frac{1}{1+x^2}\right)\, dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\left(\frac{x^3}{3} - x + \tan^{-1} x\right) + C$$

$$= \frac{1}{4}\left((x^4-1)\tan^{-1} x - (\frac{x^3}{3} - x)\right) + C$$