MH1810 Math 1 Part 4 Integration

Improper Integrals

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We know that a continuous function is Riemann integrable over a closed and bounded interval $[a, b]$ i.e., the integral $\int_a^b f(x)\, dx$ has a finite value. More generally, if f is not continuous at a finite number of points in $[a, b]$, then $\int_a^b f(x)\, dx$ exists.

However, when the integrand f is not bounded on $[a, b]$ or the interval $[a, b]$ is no longer bounded, we have to consider the corresponding integrals carefully, and the integral may or may not converge to a finite value. Such integrals are known as improper integrals.

Improper Integrals (1): Unbounded Integrand

We shall discuss two types of improper integrals.

1 Unbounded Integrand $f(x)$.
   Consider the integral
$$\int_{-1}^{1} \frac{1}{x^2}\, dx,$$
   which is not defined on $[-1,1]$, and $\dfrac{1}{x^2}$ is not bounded on $[-1,1]$ since $\displaystyle\lim_{x\to 0} \frac{1}{x^2} = \infty$. For the same reason, neither
$$\int_{-1}^{0} \frac{1}{x^2}\, dx \quad \text{nor} \quad \int_{0}^{1} \frac{1}{x^2}\, dx$$
   are defined (in the Riemann sense).

2 Unbounded Intervals.
Consider the integral $\int_1^\infty \frac{x}{1+x^2}\,dx$ and $\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$.
Both intervals $[1,\infty)$ or $(-\infty,-1]$ are not bounded.

The above integrals are called improper integrals.

How should we assign meaning to such an improper integrals?

Example

Example

Evaluate $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\, dx$.

Since $\displaystyle\lim_{x\to 0^-} \frac{1}{x^2} = \infty$, 0 is known as the **singular point.** We replace the upper limit 0 by a variable $t < 0$:

$$\int_{-1}^{t} \frac{1}{x^2}\, dx.$$

and, define the improper integral $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\, dx$ as follows:

$$\int_{-1}^{0} \frac{1}{x^2}\, dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\, dx.$$

Example

Therefore

$$\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \left(\frac{-1}{x}\right)\Bigg|_{-1}^{t}$$

$$= \lim_{t\to 0^-} \left(\frac{-1}{t} - 1\right) = \infty.$$

We say that the improper integral $\int_{-1}^{0} \frac{1}{x^2}\,dx$ diverges to infinity.

Example

Example

Evaluate $\int_0^1 \frac{1}{\sqrt{2x - x^2}}\, dx$.

Solution

Note that $\lim_{x\to 0^+} \frac{1}{\sqrt{2x - x^2}} = \infty$. *The integrand has a singular point at* 0.
Thus, we have

$\int_0^1 \frac{1}{\sqrt{2x-x^2}}\, dx = \lim_{t\to 0^+} \left(\int_t^1 \frac{1}{\sqrt{2x-x^2}}\, dx \right)$

$= \lim_{t\to 0^+} \left(\int_t^1 \frac{1}{\sqrt{1-(x-1)^2}}\, dx \right)$

$= \lim_{t\to 0^+} \left(-\sin^{-1}(t-1) \right) = -\left(\frac{-\pi}{2}\right) = \frac{\pi}{2}$.

We say that the improper integral converges to $\frac{\pi}{2}$.

Example

Example

Evaluate $\displaystyle\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx$.

Solution

Note that $\displaystyle\lim_{x\to 0^+} \frac{1}{\sqrt{2x-x^2}} = \infty$ *and* $\displaystyle\lim_{x\to 2^-} \frac{1}{\sqrt{2x-x^2}} = \infty$.
However, the function $\dfrac{1}{\sqrt{2x-x^2}}$ *is continuous on* $(0,2)$. *We evaluate the given improper integral as follows:*

$$\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx + \int_1^2 \frac{1}{\sqrt{2x-x^2}}\,dx$$

The integral $\displaystyle\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx$ *is an improper integral which is evaluated in the preceding example.*

Solution

Solution

We evaluate the improper integral $\int_1^2 \frac{1}{\sqrt{2x-x^2}}dx$ *as follows:*

$$\int_1^2 \frac{1}{\sqrt{2x-x^2}}\, dx = \lim_{t\to 2^-}\left(\int_1^t \frac{1}{\sqrt{2x-x^2}}\, dx\right)$$

$$= \lim_{t\to 2^-}\left(\int_1^t \frac{1}{\sqrt{1-(x-1)^2}}\, dx\right)$$

$$= \lim_{t\to 2^-}\left(\sin^{-1}(t-1)\right) = \frac{\pi}{2}.$$

Solution

Thus, we have

$$\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx + \int_1^2 \frac{1}{\sqrt{2x-x^2}}\,dx$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

The improper integral converges to π.

Improper Integral (2): Unbounded Intervals

Improper integrals over an unbounded interval:

$$\int_a^{\infty} f(x)\ dx,\ \int_{-\infty}^{b} f(x)\ dx,\ \int_{-\infty}^{\infty} f(x)\ dx$$

To integrate over an unbounded interval

- Replace the interval by a bounded interval, and
- Pass the definite integral to limiting process: $t \longrightarrow \infty$ or $t \longrightarrow -\infty$.

Unbounded Intervals (a)

For the improper integral $\int_a^\infty f(x)\ dx$, we define it as

$$\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx.$$

Example

Evaluate $\displaystyle\int_1^\infty \frac{1}{x^2}\, dx$.

Solution

$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = \lim_{t\to\infty} \left(- \frac{1}{x}\Bigg|_1^t \right)$

$= \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1.$

Thus, the improper integral $\int_1^\infty \frac{1}{x^2}\, dx$ converges to 1.

Unbounded Intervals (b)

For the improper integral $\int_{-\infty}^{b} f(x)\ dx$, we define it as

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_{t}^{b} f(x)\,dx.$$

Example

Evaluate $\int_{-\infty}^{-1} \frac{1}{x^3}\ dx$

Solution

$\int_{-\infty}^{-1} \frac{1}{x^3}\ dx = \lim_{t\to-\infty} \int_{t}^{-1} \frac{1}{x^3}\ dx$

$= \lim_{t\to-\infty} \left(-\frac{1}{2x^2}\Big|_{t}^{-1} \right)$

$= \lim_{t\to\infty} \left(-1/2 - \left(-\frac{1}{2t^2}\right)\right) = -1/2.$

Thus, the improper integral $\int_{-\infty}^{-1} \frac{1}{x^3}\ dx$ *converges to* $-1/2$.

Comparison Test for Integrals

Often, we want to estimate whether a given improper integral is convergent or divergent. We may use comparison theorem (stated below) to compare with a known convergent or divergent improper integrals.

Theorem (Comparison Theorem for Integrals)

Suppose f and g are continuous functions such that $f(x) \geq g(x) \geq 0$, for $x \geq a$. Then

1. *If* $\displaystyle\int_a^{\infty} f(x)\,dx$ *converges, then* $\displaystyle\int_a^{\infty} g(x)\,dx$ *converges.*
2. *If* $\displaystyle\int_a^{\infty} g(x)\,dx$ *diverges, then* $\displaystyle\int_a^{\infty} f(x)\,dx$ *diverges.*

Example

Example

Determine whether $\int_2^\infty \frac{1}{\ln x}\,dx$ converges or diverges.

Solution

For $x > 2$, *we have* $\ln x < x$ *(why?), so*

$$\frac{1}{\ln x} > \frac{1}{x}.$$

However, we have

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\left(\int_2^t \frac{1}{x}\,dx\right) = \lim_{t\to\infty}(\ln t - \ln 2) = \infty.$$

By the Comparison Theorem, we conclude that $\int_2^\infty \frac{1}{\ln x}\,dx$ *also diverges.*

The following improper integrals known as p-integrals are useful in comparison test.

Theorem (p-integrals)

Suppose $a \in \mathbb{R}$ and $a > 0$. Then

(a)

$$\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{a^{1-p}}{p-1} & \text{for } p > 1, \\ \infty & \text{for } p \leq 1. \end{cases}$$

(b)

$$\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \dfrac{a^{1-p}}{1-p} & \text{for } p < 1, \\ \infty & \text{for } p \geq 1. \end{cases}$$

Proof of (a)

Note that $\displaystyle\int_a^b \frac{1}{x^p}\,dx = \frac{b^{-p+1} - a^{-p+1}}{-p+1}$, if $p \neq 1$.

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_a^t \frac{1}{x^p}\,dx = \begin{cases} \displaystyle\lim_{t\to\infty} \frac{t^{-p+1} - a^{-p+1}}{-p+1}, & \text{if } p \neq 1 \\ \displaystyle\lim_{t\to\infty} (\ln t - \ln a), & \text{if } p = 1 \end{cases}$$

$$= \begin{cases} \dfrac{a^{1-p}}{p-1}, & \text{if } -p+1 < 0, \text{ i.e., } p > 1; \\ \infty, & \text{if } -p+1 > 0, \text{ i.e., } p \leq 1. \end{cases}$$

Proof of (b)