MH1810 Math 1 Part 4 Integration

Techniques of Integration: Rational Functions

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Integration of Rational Functions

We shall consider integrals like

$$\int \frac{x^3 + 3x^2}{x^2 + 1}\, dx, \quad \int \frac{2 + 3x + x^2}{x(x^2 + 1)}\, dx, \ldots,$$

or, in general, integrals of the form

$$\int \frac{P(x)}{Q(x)}\, dx$$

where P and Q are polynomials. How do we handle integrals like

$$\int \frac{x + 2}{x^3 - x}\, dx?$$

This is done by writing $\frac{P(x)}{Q(x)}$ as "partial fractions".

General Rational Functions

Consider rational function $\int \frac{P(x)}{Q(x)}\, dx$ where P and Q are polynomials.
Assume $\deg P < \deg Q$.
Fact from Algebra: Every polynomial

$$Q(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with real coefficient $a_0, a_1, \cdots, a_{n-1}, a_n$ can be factorized as a product of linear factors (i.e., $ax + b$) and irreducible quadratic factors (i.e., $ax^2 + bx + c$ with $b^2 - 4ac < 0$.)

Partial Fractions

The rational function $\frac{P(x)}{Q(x)}$ can thus be expressed as a sum of partial fractions given below:

factors of $Q(x)$	corresponding partial fractions
$(ax+b)^k$	$\dfrac{A_r}{(ax+b)^r}, r = 1, 2, 3, \ldots, k$
$(ax^2+bx+c)^k$	$\dfrac{A_r x + B_r}{(ax^2+bx+c)^r}, r = 1, 2, 3, \ldots, k$

Case I: Simple Linear Factors

If the denominator $Q(x)$ is a product of distinct linear (degree 1) factors (say $ax + b$), then the corresponding partial fraction representation for each factor $ax + b$ is $\dfrac{A}{ax+b}$.

Example

Express $\frac{1}{x(x-1)}$ as partial fractions.

Solution

There are two distinct linear factors x and $(x-1)$ in the denominator.
The corresponding partial fractions are $\frac{A}{x}$ and $\frac{B}{x-1}$.
If $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$, then
$\frac{1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x-A}{x(x-1)}$.
Comparing numerators, we have
$1 = (A+B)x - A$ *for all* $x \neq 0, 1$.
Thus $-A = 1$, *and* $B = 1$. *Hence* $\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$.

Example

Example

Evaluate $\int \frac{x+2}{x(x-1)(x+1)}\,dx$.

Solution

We first express $\frac{P(x)}{Q(x)} = \frac{x+2}{x(x-1)(x+1)}$ *in partial fractions.*
There are three distinct linear factors in $Q(x)$*:* x, $(x-1)$ *and* $(x+1)$.
The corresponding partial fractions are

$$\frac{A}{x},\ \frac{B}{x-1}\ \&\ \frac{C}{x+1}$$

Thus, we shall solve constants A, B *and* C *such that*

$$\frac{x+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Solution

Solution

$$\frac{x+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{A(x+1)(x-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}.$$

Comparing coefficients on the numerators:

$\diamond\ x^2 : 0 = A + B + C$

$\diamond\ x : 1 = B - C,$

$\diamond\ x^0 : 2 = -A.$

Solution

Solution

Solving this we get

$$A = -2, \quad B = 3/2, \quad C = 1/2.$$

Therefore

$$\frac{x+2}{x(x-1)(x+1)} = \frac{-2}{x} + \frac{3/2}{x-1} + \frac{1/2}{x+1}, \text{ and}$$

$$\int \frac{x+2}{x(x-1)(x+1)}\,dx = \int \left(\frac{-2}{x} + \frac{3/2}{x-1} + \frac{1/2}{x+1}\right)\,dx$$

$$= -2\int \frac{1}{x}dx + \frac{3}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx$$

Case II: Repeated Linear Factors

If $Q(x)$ is a product of linear factors, some of which are repeated, say $(ax+b)^k$, where $k \geq 2$, then there are k corresponding partial fractions:

$$\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \ldots, \frac{A_{k-1}}{(ax+b)^{k-1}}, \frac{A_k}{(ax+b)^k}$$

Example

Evaluate $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2}\,dx$.

Solution

First, we express the integrand $\frac{x^2}{(x-3)(x+2)^2}$ *in partial fractions.*

For $(x-3)$*, the corresponding partial fraction is* $\dfrac{A}{x-3}$.

For $(x+2)^2$*, the corresponding partial fractions are* $\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$.

We shall find constants A*,* B *and* C *such that*

$$\frac{x^2}{(x-3)(x+2)^2}=\frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2}.$$

Solution

Solution

$$\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2 + B(x-3)(x+2) + C(x-3)}{(x-3)(x+2)^2}.$$

Comparing coefficients:

- x^2: $1 = A + B$. *This gives* $B = 1 - A$
- x: $0 = 4A - B + C$
- x^0: $0 = 4A - 6B - 3C$

Solution

Solution

Solving for A, B *and* C *by eliminating* B *(using* $B = 1 - A$*), we have*

$$0 = 4A - (1 - A) + C \iff 5A + C = 1 - - - - - - - - - - - (1)$$

$$0 = 4A - 6(1 - A) - 3C \iff 10A - 3C = 6 - - - - - - - - - - (2)$$

(1)$\times 2$ $-$ *(2)*: $5C = -4 \iff C = -4/5$

From (1): $A = \frac{1}{5}(1 + 4/5) = 9/25$
Therefore, $B = 1 - (9/25) = 16/25.$

Solution

Solution

So, we have

$$\frac{x^2}{(x-3)(x+2)^2)} = \frac{1}{25}\left(\frac{9}{x-3} + \frac{16}{x+2} - \frac{20}{(x+2)^2}\right).$$

Thus,

$$\int \frac{x^2}{(x-3)(x+2)^2}\, dx = \frac{1}{25}\left(\int \frac{9}{x-3} + \frac{16}{x+2} - \frac{20}{(x+2)^2} dx\right)$$

$$= \frac{1}{25}\left(9\int \frac{1}{x-3}\, dx + 16\int \frac{1}{x+2} dx - 20\int \frac{1}{(x+2)^2} dx\right)$$

$$= \frac{1}{25}\left(9\ln|x-3| + 16\ln|x+2| + 20\frac{1}{(x+2)}\right) + C$$

Irreducible Quadratic Factors

The quadratic expression $ax^2 + bx + c$ is said to be irreducible when it cannot be reduced to a product of linear factors.

In this case we have $b^2 - 4ac < 0$.
The quadratic expression $ax^2 + bx + c$ can be expressed in for form $(Ax + B)^2 + D^2$, via completing square.

Case III: Distinct Irreducible Quadratic Factors

Suppose $Q(x)$ contains the quadratic factor $ax^2 + bx + c$ $(b^2 - 4ac < 0)$. Then the partial fraction representation of $\frac{P(x)}{Q(x)}$ will contain the term:

$$\frac{Ax + B}{ax^2 + bx + c}$$

A Useful Formula

Theorem

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

Example

$$\int \frac{1}{9x^2 + 25} \, dx = \frac{1}{9} \int \frac{1}{x^2 + 25/9} \, dx$$

$$= \frac{1}{9} \left(\frac{1}{5/3} \tan^{-1} \frac{x}{5/3} \right) + C = \frac{1}{15} \tan^{-1} \frac{3x}{5} + C.$$

Example

Example

Evaluate $\int \frac{1}{x^2+4x+5}\,dx$.

Solution

The quadratic expression x^2+4x+5 *is irreducible as its discriminant,* $b^2-4ac = 16-20 = -4 < 0$. *Completing square to obtain*

$$x^2+4x+5 = (x+2)^2+1.$$

Thus, we have

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx.$$

Solution

Solution

We use $u(x) = x + 2$, *with* $u'(x) = 1$.

$$\int \frac{1}{x^2 + 4x + 5}\, dx = \int \frac{1}{u^2 + 1} du$$

$$= \tan^{-1} u + C = \tan^{-1}(x + 2) + C.$$

Example

Example

Evaluate $\int \frac{1}{4x^2+4x+26}\,dx$.

Solution

Solution

$$4x^2+4x+26=(2x+1)^2+25=(2x+1)^2+5^2$$

Therefore, we have

$$\int \frac{1}{4x^2+4x+26}\,dx=\int \frac{1}{(\underbrace{2x+1}_{u})^2+5^2}\,dx=\frac{1}{2}\int \frac{1}{u^2+5^2}\,du$$

$$=\frac{1}{2}\left(\frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right)\right)+C=\frac{1}{10}\tan^{-1}\frac{2x+1}{5}+C$$

Another Useful Integral

Theorem

$\int \frac{2ax+b}{ax^2+bx+c}\ dx = \ln\left|ax^2 + bx + c\right| + C$

– from $\displaystyle\int \frac{f'(x)}{f(x)}\ dx = \ln\left|f(x)\right| + C$

Example

$$\int \frac{2x+1}{4x^2+4x+26}\ dx$$

$$= \frac{1}{4}\int \frac{8x+4}{4x^2+4x+26}\ dx = \frac{1}{4}\ln\left|4x^2+4x+26\right| + C.$$

Example

Example

Evaluate
$$\int \frac{x}{x^2+4x+13}\,dx.$$

Solution

Check that $x^2+4x+13$ *is irreducible: Its discriminant* $b^2-4ac = 4^2-4(1)(13) < 0$. *Next, completing square:*
$$x^2+4x+13 = (x+2)^2+9.$$

Solution

Solution

Now, we express

$$\frac{x}{x^2+4x+13} = \underbrace{\frac{A(2x+4)}{x^2+4x+13}}_{\frac{Af'(x)}{f(x)}} + \frac{B}{x^2+4x+13}$$

which gives $A = \frac{1}{2}$ *and* $B = -2$.

Solution

Solution

$$\int \frac{x}{x^2+4x+13}\,dx = \frac{1}{2}\underbrace{\int \frac{(2x+4)}{x^2+4x+13}\,dx}_{\ln|x^2+4x+13|+C} - 2\underbrace{\int \frac{1}{x^2+4x+13}\,dx}_{\int \frac{1}{(x+2)^2+3^2}\,dx}$$

$$= \frac{1}{2}\ln|x^2+4x+13| - 2\left(\frac{1}{3}\tan^{-1}\frac{x+2}{3}\right) + C$$

$$= \frac{1}{2}\ln|x^2+4x+13| - \frac{2}{3}\tan^{-1}\frac{x+2}{3} + C$$

Example

Example

Evaluate $\int \frac{x-1}{x^3+x}\ dx$.

Solution

The denominator $x^3 + x = x(x^2+1)$, *where* x^2+1 *is irreducible. Partial fraction representation:*

$$\frac{x-1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x)}{x(x^2+1)}$$

Comparing coefficients:
x^2: $0 = A + B$
x: $1 = C$
x^0: $-1 = A$
Solving for A, B *and* C, *we have* $A = -1, B = 1$ *and* $C = 1$.

Solution

Solution

$$\int \frac{x-1}{x^3+x}\,dx = \int -\frac{1}{x} + \frac{x+1}{x^2+1}dx = -\int \frac{1}{x}dx + \int \frac{x+1}{x^2+1}dx$$

- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C_1$
- $\displaystyle\int \frac{x+1}{x^2+1}dx = \int \frac{1}{2}\underbrace{\frac{2x}{x^2+1}}_{\ln|x^2+1|} + \underbrace{\frac{1}{x^2+1}}_{\tan^{-1}x}\,dx$

SO, we have

$$\int \frac{x-1}{x^3+x}\,dx = -\ln|x| + \frac{1}{2}\ln|x^2+1| + \tan^{-1}x + C$$

Case IV: Repeating Irreducible Factors (Optional)

Suppose $Q(x)$ contains the repeating irreducible quadratic factor $(ax^2 + bx + c)^k$. Then the partial fraction representation of $\frac{P(x)}{Q(x)}$ will contain the term:

$$\frac{A_i x + B_i}{(ax^2 + bx + c)^i}, i = 1, 2, 3, \dots, k$$

Example

Example

Evaluate $\int \frac{x^2}{x(x^2+4)^3}dx$.

Solution

Partial fractions:

$$\frac{x^2}{x(x^2+4)^3} = \frac{A}{x} + \frac{A_1x+B_1}{x^2+4} + \frac{A_2x+B_2}{(x^2+4)^2} + \frac{A_3x+B_3}{(x^2+4)^3}$$

Proceed like the above examples to solve for A_i, B_i and A, and take care of each partial fraction.

Deg of Numerator is at least Deg of Denominator

Example

$$\int \frac{x^3 + 3x^2}{x^2 + 1}\, dx.$$

Solution

Applying long division to $\dfrac{x^3 + 3x^2}{x^2 + 1}$, *we obtain that*

$$\frac{x^3 + 3x^2}{x^2 + 1} = x + 3 + \frac{-x - 3}{x^2 + 1} = x + 3 - \frac{x}{x^2 + 1} - \frac{3}{x^2 + 1}.$$

Solution

Solution

This we can integrate

$$\int \frac{x^3+3x^2}{x^2+1}\,dx = \underbrace{\int (x+3)\,dx}_{\frac{x^2}{2}+3x+C_1} - \underbrace{\int \frac{x}{x^2+1}\,dx}_{\frac{1}{2}\ln(x^2+1)+C_2} - \underbrace{\int \frac{3}{x^2+1}\,dx}_{3\tan^{-1}x+C_3}.$$

So

$$\int \frac{x^3+3x^2}{x^2+1}\,dx = \frac{x^2}{2}+3x-\frac{1}{2}\ln(x^2+1)-3\tan^{-1}x+C.$$

Solution

Solution

$$\int \frac{x^3+2}{x^3-x}\,dx = \int \left(1+\frac{x+2}{x^3-x}\right)\,dx$$

$$= x + \int \frac{x+2}{x(x-1)(x+1)}.$$

The last integral can be handled with the method of partial fractions ...