MH1810 Math 1 Part 4 Integration

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The Fundamental Theorem of Calculus

– gives the precise inverse relationship between the derivative and the definite integral.

– Newton and Leibniz exploited this relationship and used it to develop calculus into a systematic mathematical method

– for computing areas and integrals very easily without computing them as limits of sums.

Mean Value via Definite Integral

Definition

If f is continuous on $[a, b]$, then the mean value (also known as the average value) of f on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example

We have calculated $\displaystyle\int_1^3 x^2\ dx = \frac{26}{3}$. The mean value of x^2 on $[1, 3]$ is

$$\frac{1}{3-1}\cdot\frac{26}{3} = \frac{13}{3}.$$

Mean Value Theorem for Definite Integral

Theorem (The Mean Value Theorem for Definite Integrals)

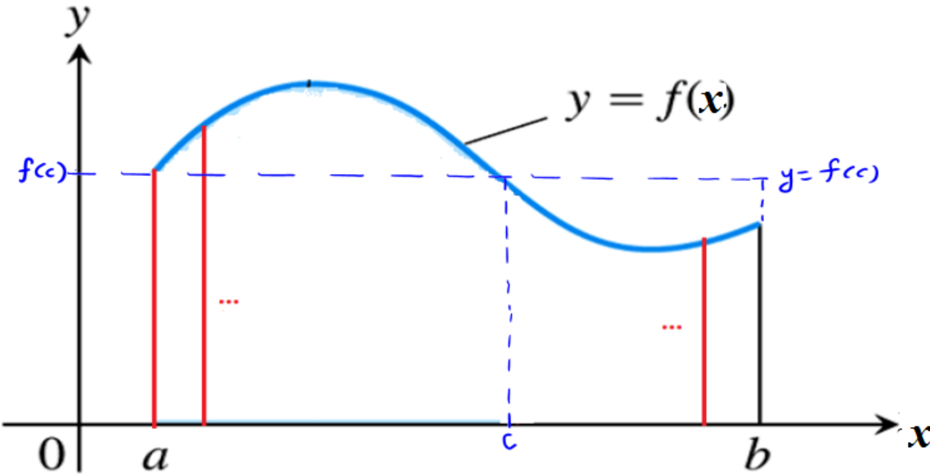
If f is continuous on $[a, b]$, then there is a point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx,$$

i.e.,

$$(b-a)f(c) = \int_a^b f(x)\,dx.$$

Suppose $f(x) > 0$.



Interpretation of the Mean Value for Definite Integral

Suppose $f(x) > 0$. The equation

$$(b-a)f(c) = \int_a^b f(x)\,dx$$

means that the area below the curve $y = f(x)$ and above the horizontal line $y = f(c)$ is equal to the area above $y = f(x)$ and below the horizontal line $y = f(c)$. So, in this sense, $f(c)$ is the average value of f on the interval $[a, b]$.

The First Fundamental Theorem of Calculus

Theorem

If f is continuous on $[a, b]$, then the function $F(x)$ defined by

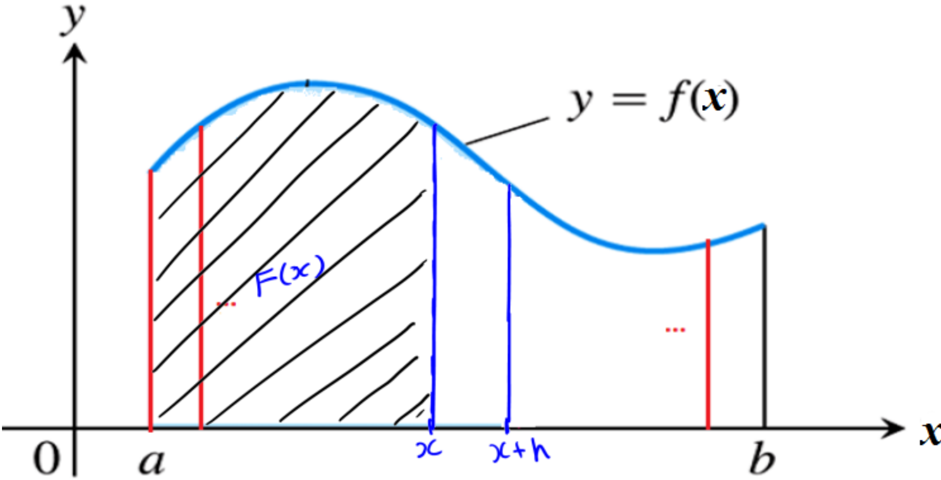
$$F(x) = \int_a^x f(t)dt, a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) and $F'(x) = f(x)$, i.e.,

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

Important Note The lower limit of integration a is a constant, and the upper limit of integration is the variable x.

For an intuitive understanding, we consider a continuous function $f(x) > 0$ on $[a, b]$.



Intuitive Idea

The definite integral $F(x) = \displaystyle\int_a^x f(t)\ dt$ is the area under $f(t)$ on the interval $[a, x]$. Note that $F(x)$ is a continuous function for $x \in [a, b]$.

$$\frac{\triangle F}{\triangle x} = \frac{F(x+h) - F(x)}{h} = \underbrace{\frac{\int_x^{x+h} f(t)\ dt}{h} = \frac{h \cdot f(x^*)}{h}}_{MVT},$$

for some $x^* \in (x, x+h)$.

As $h \to 0$, we have $x^* \to x$, thus $\dfrac{F(x+h) - F(x)}{h} \to f(x)$ (why?).

Thus, we have

$$F'(x) = f(x).$$

Example

Example

Consider $g(x) = \int_1^x \frac{\sin t}{t}\, dt$, $1 \leq x \leq b$.

By the Fundamental Theorem of Calculus, the function

$$g(x) = \int_1^x \frac{\sin t}{t}\, dt$$

is continuous on $[1, b]$ and is differentiable on $(1, b)$.

Its derivative is given by

$$g'(x) = \frac{d}{dx}\left(\int_1^x \frac{\sin t}{t}\, dt\right) = \frac{\sin x}{x}.$$

Examples

(a) $\dfrac{d}{dx}\left(\displaystyle\int_2^x (\sin t)\ln(t^2+1)\,dt\right)$

(b) $\dfrac{d}{dx}\left(\displaystyle\int_\pi^x (e^{y^2+1})\tan^3 y\,dy\right)$

(c) $\dfrac{d}{dt}\left(\displaystyle\int_{179}^t \sqrt[3]{u^4-3u+1}\,du\right)$

Example

Example

Simplify $\frac{d}{dx}\left(\int_x^{\pi} e^{(t-3)^2}\ dt\right)$.

$$\frac{d}{dx}\left(\int_x^{\pi} e^{(t-3)^2}\ dt\right) = \frac{d}{dx}\left(-\int_{\pi}^{x} e^{(t-3)^2}\ dt\right)$$

$$= -\frac{d}{dx}\left(\int_{\pi}^{x} e^{(t-3)^2}\ dt\right)$$

$$= -e^{(x-3)^2}$$

Example

Example

$\frac{d}{dx} \int_1^{\sin x} \ln(t^2 + 1)\, dt$

NOTE The Fundamental Theorem of Calculus cannot be applied directly to the function.

Solution

$\frac{d}{dx} \int_1^{\sin x} \ln(t^2 + 1)\, dt = \frac{d}{dx} \int_1^{u} \ln(t^2 + 1)\, dt$, *where* $u = \sin x$.
$= \frac{d}{du} \left(\int_1^{u} \ln(t^2 + 1)\, dt \right) \quad \frac{du}{dx}$, *(Chain Rule)*
$= \left(\ln(u^2 + 1) \right) \ \cos x$
$= (\cos x) \ln(\sin^2 x + 1)$

Theorem

$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\cdot f\left(u(x)\right).$

Proof.

The function $\displaystyle\int_a^{u(x)} f(t)\,dt$, where the upper limit of integration is a function $u(x)$ of the variable x instead of x. It is the composite function $u(x)$ followed by $\displaystyle\int_a^{x} f(t)\,dt$.
Therefore, we apply the Chain Rule. $\square$

Proof

Proof.

Let $y = u(x)$. Then

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = \frac{d}{dx}\int_a^{y} f(t)\,dt$$

$$= \frac{d}{dy}\left(\int_a^{y} f(t)\,dt\right)\frac{dy}{dx} \text{ (Chain Rule)}$$

$$= f(y)\cdot u'(x) = u'(x)\cdot f\left(u(x)\right).$$

$\square$

Example

$\frac{d}{dx}\int_0^{x^3} e^{-t^2}\,dt = e^{-(x^3)^2}\cdot(3x^2) = 3x^2 e^{-x^6}$

Example

Example

Find the first derivative of $F(x) = \int_{x^2}^{x^3} e^{-t^2}\, dt$.

Solution

We introduce a number at which the integrand e^{-t^2} *is defined. We have used the number* 0 *here.*

$$F(x) = \int_{x^2}^{x^3} e^{-t^2}\, dt = \int_0^{x^3} e^{-t^2}\, dt - \int_0^{x^2} e^{-t^2}\, dt.$$

Thus, we have

$$F'(x) = \frac{d}{dx}\left(\int_0^{x^3} e^{-t^2}\, dt\right) - \frac{d}{dx}\left(\int_0^{x^2} e^{-t^2}\, dt\right)$$

$$= 3x^2 e^{-x^6} - 2xe^{-x^4}$$

Remarks

(a) It may seem odd to have function defined via definite integral $\int_a^x f(t)\,dt$. However, such functions are not uncommon in mathematics, physics, chemistry and statistics.

(b) The Fresnel function $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$ appears in Fresnel's theory of the diffraction of light waves, and has also been applied to the design of highways.

Another example is the sine integral function $S(x) = \int_0^x \frac{\sin t}{t} dt$ in electrical engineering.

Remarks

(c) A formal way to define the natural logarithmic function is $\ln x = \int_1^x \frac{1}{t}\, dt$ for $x > 0$. From this definition, the function $\ln x$ is one-to-one, since its derivative is $1/x > 0$ for $x > 0$. Thus it has an inverse which is denoted by e^x.
From these functions, we have the formal definition of general exponential function a^x with base a defined as $a^x = e^{x \ln a}$, its inverse function is then denoted by $\log_a x$.

The Second Fundamental Theorem of Calculus

Theorem

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = G(b) - G(a),$$

where G is any antiderivative of f, i.e., $G' = f$.

The Second Fundamental Theorem of Calculus -Proof

(Follows from the First Fundamental Theorem of Calculus.)
Let $F(x) = \int_a^x f(t)\,dt$.
For $x \in (a, b)$, $F'(x) = f(x)$, by the First Fundamental Theorem of Calculus.
By assumption, $G'(x) = f(x)$. Hence,

$$G(x) = F(x) + C, \quad \text{for } x \in [a, b]$$

for some constant C, and we get

$$\begin{aligned} G(b) - G(a) = (F(b) + C) - (F(a) + C) &= F(b) - F(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt. \end{aligned}$$

The Evaluation Symbol

To facilitate evaluation of definite integrals using the Fundamental Theorem, we define

$$G(x)\Big|_a^b = G(b) - G(a),$$

which enables us to state that if f and G are continuous on $[a, b]$ and $G'(x) = f(x)$ on (a, b), then

$$\int_a^b f(x)\,dx = G(x)\Big|_a^b.$$

The Evaluation Symbol

Using the notation $\int f(x)dx$ for the indefinite integral, we may write

$$\int_a^b f(x)dx = \left(\int f(x)dx\right)\Bigg|_a^b$$

Example

Example

Note that $G(x) = \frac{x^3}{3} + \sin x + 179$ is an antiderivative of $f(x) = x^2 + \cos x$. Thus we have

$$\int_0^{\pi/2} \left(x^2 + \cos x\right)\, dx = G(\pi/2) - G(0)$$

$$= \left(\frac{(\pi/2)^3}{3} + \sin(\pi/2) + 179\right) - \left(\frac{0^3}{3} + \sin 0 + 179\right)$$

$$= \frac{\pi^3}{24} + 1.$$

Example

Example

Evaluate $\int_1^2 x^{-2} dx$.

Solution

The general antiderivative of x^{-2} *is*

$$\int x^{-2} dx = -x^{-1} + C.$$

By the Fundamental Theorem of Calculus (part 2), we have

$$\int_1^2 x^{-2} dx = \left(-x^{-1} + C\right)\Big|_1^2 = -(2)^{-1} - \left(-(1)^{-1}\right) = 1/2.$$

Example

Example

Evaluate the integral $\int_{-\pi}^{\pi} f(x)\,dx$ where

$$f(x) = \begin{cases} x & \text{if } -\pi \le x \le 0, \\ \sin x & \text{if } 0 < x \le \pi. \end{cases}$$

Solution

Note that f is continuous on $[-\pi, \pi]$.
(Exercise: Verify f is continuous at $x = 0$.)

Solution

Solution

$$f(x) = \begin{cases} x & \textit{if} \quad -\pi \le x \le 0, \\ \sin x & \textit{if} \quad 0 < x \le \pi. \end{cases}$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= \int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} \sin x\,dx$$

$$= \left(\frac{x^2}{2}\right)\Big|_{-\pi}^{0} + (-\cos x)\Big|_{0}^{\pi} = \frac{-\pi^2}{2} + 2$$

Spot the Mistake

Example

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_{-1}^{1} = -\frac{1}{1} - \left(-\frac{1}{-1}\right) = -1 - 1 = -2.$$

By any meaningful definition of the integral above, the integral should be a positive number since $1/x^2$ is always positive.

We can only use the Fundamental Theorem when the integrand f is continuous on the interval of integration.

The function $1/x^2$ is not continuous on $[-1, 1]$ so the integral above isn't actually defined (yet).

Applications

Example

Evaluate $\frac{\pi}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{n\pi}{n}\right)$.

Solution

We shall express the limit as a definite integral

$$\int_0^1 g(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\,g(x_k^*).$$

Solution

The Riemann sum of g *on* $[0,1]$ *with* x_k^* *is* $\displaystyle\sum_{k=1}^{n} \frac{1}{n}\, g(x_k^*)$.

Rewrite the sum:

$$\frac{\pi}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots + \sin\frac{n\pi}{n}\right)$$

$$= \sum_{k=1}^{n} \frac{\pi}{n}\sin\frac{k\pi}{n} = \sum_{k=1}^{n} \frac{1}{n}\, \underbrace{\pi \sin\left(\left(\frac{k}{n}\right)\pi\right)}_{g(x_k^*)}$$

Solution

Solution

Comparing

$$\sum_{k=1}^{n} \frac{1}{n}\, g(x_k^*) \text{ and } \sum_{k=1}^{n} \frac{1}{n} \underbrace{\pi \sin\left((\frac{k}{n})\pi \right)}_{g(x_k^*)},$$

we take $x_k^* = \dfrac{k}{n}$ *and* $g(x) = \pi \sin(\pi x)$ *over* $[0,1]$. *Thus, we have*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi}{n} \sin\frac{k\pi}{n} = \int_0^1 \pi \sin(\pi x)\, dx = \pi \left(\frac{-1}{\pi} \cos \pi x \right) \Big|_0^1 = 2.$$