MH1810 Math 1 Part 3 Differentiation

Nature of Extrema and Curve Sketching

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Global Extrema

Let f be a function with domain D_f. Recall

Definition

We say that f has a global maximum (respectively global minimum) at c if $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$) for all $x \in D_f$.

Our aim : find c where $f(c)$ is a global extremum (maximum or minimum).

Local (Relative) Maximum/Minimum

Definition

Let f be a function with domain D_f
(a) f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ for $x \in (u, v) \cap D_f$ where (u, v) is some open interval containing c.
(b) f has a local minimum (or relative minimum) at c if $f(c) \leq f(x)$ for $x \in (u, v) \cap D_f$ where (u, v) is some open interval containing c.

Note that a global maximum (respectively minimum) is a local maximum (respectively minimum).

Local Maximum/Minimum (Diagram)

Fermat's Theorem

We shall state, without proof, Fermat's Theorem.

Theorem (Fermat's Theorem)

Suppose f has a local maximum or minimum at c. If $f'(c)$ exists, then

$$f'(c) = 0.$$

[Proof is Omitted.]

Remark It is a useful result in locating global extrema, and plays an important role in the proving the Mean value Theorem, which is an important result in differentiation.

Classifying Local Extrema

We know that if $f(c)$ is a local extremum , then c is a critical point.

We search for critical points: finding c at which $f'(c) = 0$ or $f'(c)$ is not defined.

Now, if given a critical point, it is often useful to know the nature of the critical point,
i.e., can we tell whether $f(c)$ is a local maximum or local minimum or neither?

The First Derivative Test

Theorem

Suppose that f is continuous in a neighbourhood of c where c is a critical point of f and that f' exists in a deleted neighbourhood of c. (Note that $f'(c)$ may not be defined.)
(a) If $f'(x)$ changes from negative to positive as x increases through c, then f has a local minimum at c.

The First Derivative Test

Theorem

Suppose that f is continuous in a neighbourhood of c where c is a critical point of f and that f' exists in a deleted neighbourhood of c. (Note that $f'(c)$ may not be defined.)

(b) If $f'(x)$ changes from positive to negative as x increases through c, then f has a local maximum at c.

(c) If $f'(x)$ does not change sign as x increases through c, then f has no maximum or minimum at c.

Example

Example

Let $f(x) = (x-1)^{2/3}$. Find and classify all critical points of f on $\mathbb{R}$.

Solution

We have

$$f'(x) = \frac{2}{3}(x-1)^{-1/3},$$

which is undefined at $x = 1$. Hence we have a singular point at $x = 1$. Furthermore, since $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$, the first derivative test tells us that $f(1) = 0$ is a local minimum for f.

Example

Example

Let $f(x) = (x-1)^{1/3}$. Find and classify all critical points of f on $\mathbb{R}$.

Solution

We have

$$f'(x) = \frac{1}{3}(x-1)^{-2/3},$$

which is undefined at $x = 1$. Hence we have a singular point at $x = 1$. Furthermore, $f'(x) > 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$, so the first derivative test tells us that $f(1) = 0$ is neither a local maximum nor a local minimum for f.

The First Derivative Test

Theorem

Suppose $f'(c) = 0$ and f'' is continuous near c.

(a) *If $f''(c) > 0$, then f has a local minimum at c.*
(b) *If $f''(c) < 0$, then f has a local maximum at c.*
(c) *If $f''(c) = 0$, there is no conclusion. We don't know whether f has a local maximum or local minimum at c.*

Graphical explanation:

Example

Example

Let $f(x) = 2 + 3x - x^3$. Classify all critical points of f.

Solution

$f(x) = 2 + 3x - x^3$, $f'(x) = 3 - 3x^2$,$f''(x) = -6x$ *at every* $x \in \mathbb{R}$.
Critical points are $x = 1$ *and* $x = -1$.
At $x = 1$, *note that* $f'(1) = 0$ *and* $f''(1) < 0$. *By the second derivative test,* f *has a local maximum at* $x = 1$.
At $x = -1$, *note that* $f'(-1) = 0$ *and* $f''(1) > 0$. *By the second derivative test,* f *has a local minimum at* $x = -1$.

Example

Example

Classify all critical points of $f(x) = x^4$.

Solution

$f(x) = x^4$, $f'(x) = 4x^3$,$f''(x) = 12x^2$ at every $x \in \mathbb{R}$.
It is clear that $x = 0$ is the only critical point.
The second derivative test can not be applied here as $f''(0) = 0$. We shall use first derivative test.
For $x < 0$, $f'(x) < 0$ whereas $f'(x) > 0$ for $x > 0$. By the first derivative test, we conclude f has a local minimum at $x = 0$.

Curve Sketching

From what we have discussed in this chapter, we can obtain useful information about the shape of graph of a function and proceed to sketch the graph of a function.

Some useful steps.

(a) Find the interval(s) of increase or decrease.

(b) Find the interval(s) of concavity (i.e., when will the function concave upward/downward?)

(c) Identify local extrema and point of inflection.

Curve Sketching - Some Useful Steps

(d) Find all vertical asymptotes $x = a$.
(i.e., Find a such that $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.)

(e) Find all horizontal asymptotes $y = b$.
(i.e., Find b such that $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.)

(f) Use the information to sketch the graph of $y = f(x)$.

Example

Example

Sketch the graph of $y = 2 + 3x - x^3$.

Solution

Let $f(x) = 2 + 3x - x^3$. *Then* $f'(x) = 3 - 3x^2$ *and* $f''(x) = -6x$ *at every* $x \in \mathbb{R}$.

Interval of increase/decrease. $f'(x) = 3 - 3x^2 = 3(1 - x)(1 + x)$ *on* $\mathbb{R}$.

Thus, $f'(x) > 0$ *for* $x \in (-1, 1)$ *and* $f'(x) > 0$ *for* $x \in (-\infty, -1) \cup (1, \infty)$.

Since f *is continuous* $\mathbb{R}$, *we conclude that* f *is increasing on* $[-1, 1]$.

Solution

Concavity. *Since* $f''(x) = -6x$,
$f''(x) > 0 \iff x < 0$, *and* $f''(x) < 0 \iff x > 0$.
Therefore, the graph of f *is concave downward on* $(0, \infty)$, *and concave upward on* $(-\infty, 0)$.
There is a change of concavity at $x = 0$.
So, $x = 0$ *is a point of inflection.*

Solution

Max/Min. *Since* $f'(x) = 3 - 3x^2$, $f''(x) = -6x$.
Critical points are $x = 1$ *and* $x = -1$.
At $x = 1$, *note that* $f'(1) = 0$ *and* $f''(1) < 0$.
By the second derivative test, f *has a local maximum at* $x = 1$.
At $x = -1$, *note that* $f'(-1) = 0$ *and* $f''(-1) > 0$. *By the second derivative test,* f *has a local minimum at* $x = -1$.

Solution

Asymptotes.

$f(x)$ is continuous at every real number. So, there is no vertical asymptote.

Next we have $\lim_{x \to \infty} f(x) = -\infty$ *and* $\lim_{x \to -\infty} f(x) = \infty$.

Thus, there is no horizontal asymptote.