MH1810 Math 1 Part 3 Differentiation

First Derivative and Growth of a Function

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First Derivative and the Growth of a Function

Suppose f is differentiable and f is increasing on (a, b). Then it follows from the definition of derivative that $f'(x) \geq 0$ on (a, b).

How about the converse?

If $f'(x) \geq 0$ on (a, b), does it follow that f is increasing on (a, b)?

The next result says that it is true if $f'(x) > 0$.

Theorem

Theorem

1. *If* $f'(x) > 0$ *on* (a, b)*, then* f *is increasing on* (a, b)*, i.e., for* $x_1, x_2 \in (a, b)$
$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$
2. *If* $f'(x) < 0$ *on* (a, b)*, then* f *is decreasing on* I*, i.e., for* $x_1, x_2 \in (a, b)$
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

Proof.

(Use Mean Value Theorem) $\square$

Corollary

Suppose f *continuous on* $[a, b]$.

1. *If* $f'(x) > 0$ *on* (a, b), *then* f *is increasing on* $[a, b]$.
2. *If* $f'(x) < 0$ *on* (a, b), *then* f *is decreasing on* $[a, b]$.

Example

Example

Find interval(s) where f defined by $f(x) = 2 + 3x - x^3$ is increasing.

Solution

Solution

The function $f(x) = 2 + 3x - x^3$ *is continuous on* $\mathbb{R}$.
Note that

$$f'(x) = 3 - 3x^2 = 3(1-x)(1+x) \text{ on } \mathbb{R}.$$

Thus, $f'(x) > 0$ *for* $x \in (-1, 1)$ *and* $f'(x) < 0$ *for* $x \in (-\infty, -1) \cup (1, \infty)$.
Since f *is continuous* $\mathbb{R}$, *we conclude that* f *is increasing on* $[-1, 1]$.

Using f' for checking one-to-one

If f is increasing or decreasing on (a, b), then f is one-to-one on (a, b).

Example

Show that $f(x) = \sin x$ with domain $[-\pi/2, \pi/2]$ is one-to-one.

Solution

We have

$$f'(x) = \cos x > 0, \quad x \in (-\pi/2, \pi, 2).$$

So, f is continuous on $[-\pi/2, \pi/2]$, differentiable on $(-\pi/2, \pi, 2)$ and $f'(x) > 0$ on $(-\pi/2, \pi, 2)$.
Thus, f is increasing on $[-\pi/2, \pi/2]$ and hence it is one-to-one. (And, its inverse is denoted by $\sin^{-1} x$.)

Example

How do we construct a cylindrical metal can with a given volume V in a way that minimizes the surface area (the amount of metal used)?

Solution

Let r be the radius and h the height and A the surface area of the cylindrical can. Then

$$V = \pi r^2 h, \quad A = 2\pi r^2 + 2\pi rh.$$

The first equation gives us $h = \dfrac{V}{\pi r^2}$, which we can substitute into A to get $A(r) = 2\pi r^2 + \dfrac{2V}{r}$.

Solution

$$A(r) = 2\pi r^2 + \frac{2V}{r}.$$

Our objective is to find the minimum of $A(r)$*, where the domain of* $A(r)$ *is* $(0, \infty)$*.Note that* A *is continuous on* $(0, \infty)$*, and we have*

$$A'(r) = 4\pi r - \frac{2V}{r^2} = \frac{4\pi}{r^2}\left(r^3 - \frac{V}{2\pi}\right).$$

Solution

Solution

$A'(r) = \frac{4\pi}{r^2}\left(r^3 - \frac{V}{2\pi}\right) = 0 \Leftrightarrow r = \left(\frac{V}{2\pi}\right)^{1/3}$

For $0 < r < \left(\frac{V}{2\pi}\right)^{1/3}$, $A'(r) < 0$. *Thus,* $A(r)$ *is decreasing on* $(0, \left(\frac{V}{2\pi}\right)^{1/3})$.

For $r > \left(\frac{V}{2\pi}\right)^{1/3}$, $A'(r) > 0$. *Thus,* $A(r)$ *is increasing on* $(\left(\frac{V}{2\pi}\right)^{1/3}, \infty)$.

Therefore, $A(r)$ *where* $r = \left(\frac{V}{2\pi}\right)^{1/3}$ *must be a global minimum point.*

Hence we should choose to make our cans with radius $r = \left(\frac{V}{2\pi}\right)^{1/3}$ *and height* $h = V/(\pi r^2)$.

Second Derivatives and Shape of Curve

We start with describing the shape of a curve, followed by using the second derivative to classify its shape.

Definition

Suppose f is differentiable.

(a) The graph of a function f concaves upward at a point c if the graph of f lies above its tangent at c, i.e.,

$$f(x) \geq f(c) + f'(c)(x - c)$$

for x in a neighbourhood of c.
The graph of a function f concaves upward on an interval (a, b) if it is concave upward (or convex) at every point in (a, b).

Definition

Suppose f is differentiable.

(b) The graph of a function f concaves downward at a point c if the graph of f lies below its tangent at c, i.e.,

$$f(x) \leq f(c) + f'(c)(x - c)$$

for x in a neighbourhood of c.
The graph of a function f concaves downward on an interval (a, b) if it is concave downward (or concave) at every point in (a, b).

Inflection Points

Definition

Suppose f is differentiable.

(c) A point P on the curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes concavity, i.e., from concaving upward to concaving downward, or from concaving downward to concaving upward.

Concavity Test

Theorem

(a) *If* $f''(x) \geq 0$ *for all* x *in* (a, b), *then the graph of* f *concaves upward on* (a, b).

(b) *If* $f''(x) \leq 0$ *for all* x *in* (a, b), *then the graph of* f *concaves downward on* (a, b).

[Proof] OMITTED.
This is a consequence of the Mean Value Theorem applied to f'.

Example

Let $f(x) = 2 + 3x - x^3$. Find the intervals where the graph concave upwards. Find also the intervals where the graph concaves downwards and the points of inflection.

Solution

$f(x) = 2 + 3x - x^3$, $f'(x) = 3 - 3x^2$, $f''(x) = -6x$ *at every* $x \in \mathbb{R}$.

$$f''(x) > 0 \iff x < 0,$$

$$f''(x) < 0 \iff x > 0.$$

Therefore, the graph of f is concave downward on $(0, \infty)$, and concave upward on $(-\infty, 0)$.

There is a change of concavity at $x = 0$. So, $x = 0$ is a point of inflection.

Second derivatives and the nature of extrema

The next result is useful for solving some optimization problems, especially if the function is twice differentiable.

Theorem

Suppose f is twice differentiable on (a, b) and $f'(c) = 0$ for some $c \in (a, b)$.
(a) If $f''(x) \geq 0$ on (a, b), then $f(c)$ is a global minimum.
(b) If $f''(x) \leq 0$ on (a, b), then $f(c)$ is a global maximum.

[Proof.] Omitted.

Application to an Optimisation Problem

Example

How do we construct a cylindrical metal can with a given volume V in a way that minimizes the surface area (the amount of metal used)?

Solution

Let r be the radius and h the height and A the surface area of the cylindrical can. Then

$$V = \pi r^2 h, \text{ and } A = 2\pi r^2 + 2\pi rh.$$

The first equation gives us $h = \dfrac{V}{\pi r^2}$, which we can substitute into A to get

$$A(r) = 2\pi r^2 + \frac{2V}{r}.$$

Solution

Note that $A(r)$ is continuous on $(0, \infty)$, and we have

$$A'(r) = \frac{4\pi}{r^2}\left(r^3 - \frac{V}{2\pi}\right) \text{ and } A''(r) = 4\pi + \frac{4V}{r^2} > 0.$$

We also note that

$$A'(r) = 0 \Leftrightarrow r = \left(\frac{V}{2\pi}\right)^{1/3}$$

Since $A''(r) > 0$ for every $r \in (0, \infty)$ and $A'\left(\left(\frac{V}{2\pi}\right)^{1/3}\right) = 0$, we conclude that $A\left(\left(\frac{V}{2\pi}\right)^{1/3}\right)$ is a global minimum.