MH1810 Math 1 Part 3 Differentiation

Closed Interval Method

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Optimization Problem

– problems aiming to find global extreme values
– very practical and important in many different areas

- Finding the shortest time or shortest path or least cost in a transportation problem;
- Fermat's Principle in optics: Light follows path that takes the least time;
- Finding the least material required to construct something subject to some constraints;
- Obtaining the maximum profit to produce a commodity;
- Constructing cylindrical metal can with a given volume V in a way that minimizes the surface area (the amount of metal used).

Closed Interval Method

In this section, we discuss the closed interval method to solve an optimization problem where the function involved is continuous and the domain is a closed and bounded interval.

Critical Points

Critical points are points c at which $f'(c) = 0$ or $f'(c)$ fails to exist.

1. A point c where $f'(c) = 0$ is called a stationary point.
2. A point where $f'(c)$ fails to exist is called a singular point.

Closed Interval Method

Recall that the Extreme Value Theorem states that a continuous function f on a closed and bounded interval $[a, b]$ attains its global maximum and global minimum.

The following three-step procedure can be used to find global maximum and absolute minimum of a continuous function f on a closed and bounded interval $[a, b]$.

Closed Interval Method

1: Determine all critical points of f in (a, b) and find the corresponding f-values.

2: Compute $f(a)$ and $f(b)$.

3: The largest (respectively smallest) value of f from Steps 1 and 2 is the global maximum (respectively global minimum) of f on $[a, b]$.

Example

Example

Find the global maximum and global minimum of $f(t) = \sqrt[3]{t}\,(8-t)$ on $[-1, 8]$.

Solution

Note that f is continuous on $[-1, 8]$, since it is the product function of continuous functions $\sqrt[3]{t}$ and $8 - t$. By the Extreme Value Theorem, f has a global minimum on $[-1, 8]$.

Solution

Example

Find the global maximum and global minimum of $f(t) = \sqrt[3]{t}\,(8-t)$ on $[-1, 8]$.

Solution

To find critical points of f, we have to find c at which $f'(c)$ does not exist or $f'(c) = 0$.

For $-1 < t < 0$ or $0 < t < 8$, we have

$f'(t) = \frac{1}{3}t^{-2/3}(8-t) - \sqrt[3]{t} = \frac{8-4t}{3(\sqrt[3]{t})^2}$.

Singular point: $t = 0$ *is a singular point of f, since f is not differentiable at $t = 0$.*

Stationary Point: $f'(t) = 0 \iff 8 - 4t = 0 \iff t = 2$.

End Points: $t = -1$ *and* $t = 8$.

Solution

Comparing values of f:

$$f(2) = \sqrt[3]{2}(6), \quad f(0) = 0, \quad f(-1) = -9, \quad f(8) = 0.$$

Conclusion: Global maximum of f on $[-1, 8]$ is $f(2) = \sqrt[3]{2}(6)$
Global minimum of f on $[-1, 8]$ is $f(-1) = -9$.

Example

Example

Let $f(x) = \left(x^2 - 1\right)^{2/3}$. Find the global maximum and global minimum values of f on the interval $[-3, 3]$

Solution

The function f is continuous on $[-3, 3]$.

By the Extreme Value Theorem, it has a global maximum and a global minimum. We have

$$f'(x) = \frac{2}{3}(x^2 - 1)^{-1/3} \cdot 2x = \frac{4x}{3(x^2 - 1)^{1/3}}.$$

Solution

Solution

We have $f'(x) = \dfrac{4x}{3(x^2-1)^{1/3}}$.

Critical points:

– Stationary point: $f'(x) = 0 \quad \Leftrightarrow \quad x = 0$ and $f(0) = 1$.
– Singular points: $f'(x)$ fails to exist when $x = \pm 1$ and
$f(-1) = 0, \quad f(1) = 0$.

Endpoints: $f(-3) = 4, \quad f(3) = 4$

Since these are all candidates for extreme values, we see that the largest value of f on $[-3, 3]$ is $f(-3) = f(3) = 4$ and the smallest value is $f(-1) = f(1) = 0$.