MH1810 Math 1 Part 3 Differentiation

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Theorem

If a function f is differentiable at $x = c$ then f is continuous at $x = c$.

Proof.

Suppose f is differentiable at $x = c$. This means that f is defined on some open interval containing c and the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

exists. Consequently, we have □

Differentiability and Continuity

Proof.

$$\lim_{x\to c} f(x) = \lim_{x\to c} (f(x) - f(c) + f(c)) = \lim_{x\to c} \left(\frac{f(x) - f(c)}{x - c}(x - c) + f(c) \right)$$

$$= \lim_{x\to c} \left(\frac{f(x) - f(c)}{x - c}(x - c) \right) + \lim_{x\to c} (f(c))$$

$$= \lim_{x\to c} \left(\frac{f(x) - f(c)}{x - c} \right) \lim_{x\to c} (x - c) + f(c) = f(c).$$

Hence, f is continuous at $x = c$. $\square$

Differentiability and Continuity

However, it is possible for a function to be continuous at $x = c$ but not differentiable at $x = c$.
For example $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Differentiability and Continuity

As a consequence of the theorem, we have

Corollary

If f is not continuous at $x = c$, then f is not differentiable at $x = c$.

Differentiability and Continuity

The following result is useful to determine derivative of a piecewise defined function f which 'splits' at $x = c$.

Theorem

Suppose f is continuous at $x = c$. If

$$\lim_{x \to c^+} f'(x) = \lim_{x \to c^-} f'(x) = L,$$

then f is differentiable at $x = c$ and $f'(c) = L$.

Proof.

(Omitted) □

Example

Example

Let $f(x) = \begin{cases} x^3 + 2 & \text{if } x > 1 \\ 3x & \text{if } x \leq 1 \end{cases}$, find $f'(x)$.

Solution

Note that

- $f(1) = 3$,
- $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^3 + 2) = 3$ *and*
- $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x) = 3.$

Thus, $\lim_{x \to 1} f(x) = 3 = f(1)$, *which says that f is continuous at $x = 1$.*

Solution

For $x > 1$ we have $f'(x) = 3x^2$ whereas $f'(x) = 3$ for $x < 1$.

Since f is continuous at $x = 1$ and $\lim_{x \to 1^+} f'(x) = 3 = \lim_{x \to 1^-} f'(x)$, the function f is differentiable at $x = 1$ and $f'(1) = 3$.
In conclusion, we have

$$f'(x) = \begin{cases} 3x^2 & \text{if } x > 1 \\ 3x & \text{if } x \leq 1 \end{cases}$$

Differentiation Rules

Suppose f and g are differentiable at $x = c$. Then

(1) $f + g$ is differentiable at $x = c$ and

$$\text{(Sum Rule) } (f+g)'(c) = f'(c) + g'(c).$$

(2) $f - g$ is differentiable at $x = c$ and

$$\text{(Difference Rule) } (f-g)'(c) = f'(c) - g'(c).$$

Proof of Sum Rule

INDEPENDENT READING.

$$
\begin{aligned}
&\lim_{h \to 0} \frac{(f+g)(c+h) - (f+g)(c)}{h} \\
&\qquad = \lim_{h \to 0} \frac{(f(c+h) + g(c+h)) - (f(c) + g(c))}{h} \\
&= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} + \lim_{h \to 0} \frac{g(c+h) - g(c)}{h} = f'(c) + g'(c)
\end{aligned}
$$

Difference Rule: Similar to proof of SUM RULE.

(3) fg is differentiable at $x = c$ and

$$\text{(Product Rule)}\ (fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

(4) f/g is differentiable at $x = c$, provided $g(c) \neq 0$ in a neighbourhood of c, and

$$\text{(Quotient Rule)}\ \left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

(5) Suppose $g(x) \neq 0$ in a small neighbourhood of c. The function $\dfrac{1}{g}$ is differentiable at $x = c$, and

$$\text{(Reciprocal Rule)}\ \left(\frac{1}{g}\right)'(c) = \frac{-g'(c)}{\left(g(c)\right)^2}.$$

Proof of Product Rule

$$\lim_{h\to 0}\frac{(fg)(c+h)-(fg)(c)}{h}=\lim_{h\to 0}\frac{f(c+h)g(c+h)-f(c)g(c)}{h}$$

$$=\lim_{h\to 0}\frac{f(c+h)g(c+h)-f(c+h)g(c)+f(c+h)g(c)-f(c)g(c)}{h}$$

$$=\lim_{h\to 0}f(c+h)\frac{g(c+h)-g(c)}{h}+\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}g(c)$$

Since f is differentiable at $x=c$ it is continuous at $x=c$ and so $\lim_{h\to 0}f(c+h)=f(c)$ and we get

$$\lim_{h\to 0}\frac{(fg)(c+h)-(fg)(c)}{h}=f(c)g'(c)+f'(c)g(c).$$

Proof of Quotient and Reciprocal Rules

(Exercise or Check with the textbook.)

Derivative of a linear combination

(6) The function $\alpha f + \beta g$ is differentiable at $x = c$, where α and β are real constants, and

$$(\alpha f + \beta g)'(c) = \alpha f'(c) + \beta g'(c).$$

Power Rule

Using the results established above, we can prove the power rule for negative exponents.

Proposition (The Power Rule)

Suppose m be a negative integer, say $m = -n$, where $n \in \mathbb{Z}^{+}$. If $f(x) = x^{m}$ where $x \neq 0$, then $f'(x) = mx^{m-1}$.

Proof of Power Rule (Independent Reading)

We have proved that $f'(x) = \frac{dx^n}{dx} = nx^{n-1}$ for positive integer n.
By reciprocal rule, we have

$$f'(x) = \frac{dx^m}{dx} = \frac{d}{dx}\left(x^{-n}\right) = \frac{d}{dx}\left(\frac{1}{x^n}\right)$$

$$= \frac{-\frac{dx^n}{dx}}{(x^n)^2} = \frac{(-n)x^{n-1}}{x^{2n}} = (-n)x^{(-n)-1} = mx^{m-1}.$$

In conclusion, we have

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}, \text{ for every (positive and negative) integer } n.$$

Chain Rule

Theorem (The Chain Rule)

Let f and g be functions such that $g(x)$ is differentiable at $x = c$ and $f(u)$ is differentiable at $u = g(c)$. Then $f \circ g$ is differentiable at $x = c$ and

$$(f \circ g)'(c) = f'(g(c))g'(c).$$

Or with Leibniz notation

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}.$$

[Proof is omitted.] A correct proof of the chain rule is not in the syllabus.

Example

Example

Differentiate $h(x) = \sqrt{3x^5 - 6x^2 + 7x + \pi}$.

Solution

Let $f(x) = \sqrt{x}$ *and* $g(x) = 3x^5 - 6x^2 + 7x + \pi$. *Then* $h = f \circ g$. *By chain rule,* $h'(x) = f'(g(x))\, g'(x)$ *where*

$$f'(x) = \frac{1}{2\sqrt{x}} \ \& \ g'(x) = 15x^4 - 12x + 7.$$

$$f'(g(x)) = \frac{1}{2\sqrt{g(x)}} = \frac{1}{2\sqrt{3x^5 - 6x^2 + 7x + \pi}}.$$

$$h'(x) = \frac{15x^4 - 12x + 7}{2\sqrt{3x^5 - 6x^2 + 7x + \pi}}.$$

Consequences of the Chain Rule

It follows from the Chain rule that

Corollary

(a) $\dfrac{d}{dx}\left(f(x)\right)^n = n\left(f(x)\right)^{n-1} f'(x)$, *where* $n \in \mathbb{Z}$.

(b) $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{1}{2}\dfrac{f'(x)}{\sqrt{f(x)}}$, *where* $f(x) > 0$.

Derivatives of Trigonometric Functions

Theorem

1. $$\frac{d}{dx}(\sin x) = \cos x$$
2. $$\frac{d}{dx}(\cos x) = -\sin x$$
3. $$\frac{d}{dx}(\tan x) = \sec^2 x$$
4. $$\frac{d}{dx}(\sec x) = \sec x \tan x$$
5. $$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
6. $$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Trigonometric Functions

Using chain rule, we have

Corollary

1. $\dfrac{d}{dx}\sin(Ax+B) = A\cos(Ax+B)$
2. $\dfrac{d}{dx}\cos(Ax+B) = -A\sin(Ax+B)$
3. $\dfrac{d}{dx}\tan(Ax+B) = A\sec^2(Ax+B)$
4. $\dfrac{d}{dx}\sin\left(f(x)\right) = f'(x)\cos\left(f(x)\right)$

Proof of derivative sin(x) is cos(x)

Two results are used -

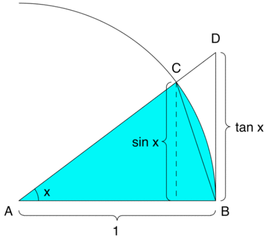
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

and

Proposition

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

Proof (Independent Reading)



- Assume $x > 0$. Arc $BC = x$.
- Therefore from the diagram, $\sin x \leq x$ and $x \leq \tan x$.
- Thus $\dfrac{\sin x}{x} \leq 1$ and $\cos x \leq \dfrac{\sin x}{x}$.
- Thus $\cos x \leq \dfrac{\sin x}{x} \leq 1$.
- Since $\lim_{x \to 0^+} \cos x = 1$,
- by squeez theorem, $\lim_{x \to 0^+} \dfrac{\sin x}{x} = 1$.

Proof (Independent Reading)

For $x \neq 0$ we have,

$$\frac{\cos x - 1}{x} = \frac{\big((1 - 2\sin^2(x/2)) - 1\big)}{x} = \frac{-2\sin^2 x/2}{x}$$

$$= -\underbrace{\frac{\sin x/2}{x/2}}_{\to 1} \cdot \underbrace{\sin x/2}_{\to 0} \to -1 \cdot 0 = 0, \quad \text{as } x \to 0.$$

Proof of derivative sin(x) is cos(x)

With $f(x) = \sin x$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$$

$$= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \sin x \underbrace{\frac{(\cos\ \ h - 1)}{h}}_{\to 0} + \cos x \underbrace{\frac{\sin h}{h}}_{\to 1}$$

$$\to \sin x \cdot 0 + \cos x \cdot 1 = \cos x, \quad \text{as } h \to 0.$$

Proof of the rest of trigonometric derivatives

Once we have established derivative of the sine function, we may apply rules of differentiation to derive the derivative of other trigonometric functions. (Exercise.)

Exponential and Logarithmic Functions

Theorem

1. $\frac{d}{dx}(e^x) = e^x$
2. $\frac{d}{dx}(a^x) = a^x \ln a,\ a > 0 \ \&\ a \neq 1.$
3. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
4. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a},\ a > 0 \ \&\ a \neq 1.$

It's important to note that the exponential function e^x (also denoted by $\exp(x)$) is the only function such that $f'(x) = f(x)$.

Examples

(a) $$\frac{d}{dx}\left(x^5+\sin(4x)-xe^x\right)=5x^4+4\cos(4x)-e^x-xe^x.$$

(b) $$\frac{d}{dx}\left(\frac{e^x}{3x^3-x+1}\right)=\frac{e^x(3x^3-x+1)-e^x(9x^2-1)}{(3x^3-x+1)^2}$$
$$=\frac{e^x(3x^3-9x^2-x+2)}{(3x^3-x+1)^2}$$

(c) $$\frac{d}{dx}\left(\tan(\sqrt{x})\right)=\sec^2(\sqrt{x})\frac{1}{2\sqrt{x}}=\frac{\sec^2(\sqrt{x})}{2\sqrt{x}}$$

(d) $$\frac{d}{dx}\left(\ln(\cos^2 x+1)\right)$$
$$=(-2\cos x\sin x)\frac{1}{\cos^2 x+1}=\frac{-2\cos x\sin x}{\cos^2 x+1}$$

Further Results

Proposition

(a) $\dfrac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x)$

(b) $\dfrac{d}{dx}\ln\ f(x) = \dfrac{f'(x)}{f(x)},\ where\ f(x) > 0$

[Proof – by Chain rule.]