MH1810 Math 1 Part 2 Limits and Continuity

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Continuous Functions

Very often, we are interested in functions which are continuous on an intervals, or disjoint union of intervals.
We shall discuss two useful results related to a function which is continuous on a closed and bounded interval $[a, b]$.

- Intermediate Value Theorem;
- Extreme Value Theorem.

Intervals and End-points

Consider intervals

$$(a, b), [a, b), (a, b], [a, b], (a, \infty), [a, \infty), (-\infty, b), (-\infty, b].$$

The points a and b are called endpoints.
The point a is a left endpoint and b is a right endpoint.
A point x in an interval is called an interior point of the interval if it is not an endpoint.

Continuous on Interval

Definition

The function f is said to be continuous on the interval I if

1. f is continuous at every interior point c of I, i.e., $\lim_{x \longrightarrow c} f(x) = f(c)$.
2. If the left endpoint a of I is included in I, f is continuous from the right there, i.e., $\lim_{x \longrightarrow a^+} f(x) = f(a)$.
3. If the right endpoint b of I is included in I, f is continuous from the left there, i.e., $\lim_{x \longrightarrow b^-} f(x) = f(b)$.

Graph of a Continuous Function on an interval

– a curve which has no break and no jump.

Known Continuous Functions

Some known functions which are continuous on their domains.

- polynomials
- *n*-root functions
- trigonometric functions
- exponential functions and logarithmic functions
- hyperbolic functions

For the above functions, to determine $\lim_{x \to c} f(x)$ limit at a point $x = c$, we can simply substitute the value c into the function.

Heaviside Function

Example

The Heaviside function

$$H(x) = \begin{cases} 0 & ; \ forx < 0, \\ 1 & ; \ forx \geq 0. \end{cases}$$

is continuous on the interval $[0, \infty)$, but not on $(-\infty, \infty)$ and not on $(-\infty, 0]$.

Combination of Continuous Functions

Theorem

Suppose functions f and g are continuous on a set S. Then the following combinations of functions

$$f \pm g, f \cdot g, f/g(\text{ provided } g(c) \neq 0)$$

are also continuous on S.

Example

The function f defined by $f(x) = \dfrac{\sin(x - e^x)}{\ln x}$ is continuous on $(1, \infty)$.

Composite of Continuous Functions

Proposition

Suppose the function f is continuous on its domain D_f and the function g is continuous on the range R_f of f. Then the composite function $g \circ f$ is continuous on D_f.

Example

Consider $f(x) = x^3$ and $g(x) = \sin x$ which are continuous on $\mathbb{R}$. Their composite functions $f \circ g(x) = (\sin x)^3$ and $g \circ f(x) = \sin(x^3)$ are also continuous on $\mathbb{R}$.

Example (a): Composite Function

Example

Let $f(x) = \sqrt{x-4}$ on $[4, \infty)$ and $g(x) = x^2$.
Find the composite function $g \circ f$ and its the domain where $g \circ f$ is continuous on.

Solution

Since f is continuous on $[4, \infty)$ and $g(x) = x^2$ is continuous on $\mathbb{R}$, the composite function $g \circ f$ is continuous on the domain on which $(g \circ f)(x)$ is defined.

Example (a): Composite Function

Solution (cont'd)

To find this domain, we note that $g \circ f(x) = g(f(x))$

$$= g\left(\sqrt{x-4}\right) \text{ provided } x - 4 \geq 0, \text{ i.e., } x \geq 4$$

$$= \left(\sqrt{x-4}\right)^2 = x - 4 \text{ provided } x \geq 4$$

Thus, we have $(g \circ f)(x) = x - 4$ *and its domain is* $[4, \infty)$.
We conclude that $(g \circ f)(x) = x - 4$ *is continuous on* $[4, \infty)$.

Note

Although the composite function $g \circ f$ and $h(x) = x - 4$ have the same expression, they are considered different functions because $(g \circ f)(x) = x - 4$ is only defined on $[4, \infty)$ while $h(x) = x - 4$ holds for every x in $\mathbb{R}$.

Example (b): Composite Function

Example

Let $f(x) = \sqrt{x-4}$ on $[4, \infty)$ and $g(x) = x^2$.
Find the composite function $f \circ g$ and its the domain where $f \circ g$ is continuous on.

Solution

To find the domain and $f \circ g$, we proceed to determine $(f \circ g)(x)$ as follows:

$$(f \circ g)(x) = f\left(g(x)\right).$$

Example (b): Composite Function

Solution (cont'd)

To determine $(f \circ g)(x)$ *as follows:*

$$(f \circ)g(x) = f\,(g(x)) = f(x^2), \text{ which holds for every real number } x,$$

$$= \sqrt{x^2 - 4}, \text{ which holds provided } x^2 \geq 4.$$

Thus, we have $(f \circ g)(x) = \sqrt{x^2 - 4}$ *where* $x \geq 2$ *or* $x \leq 2$*. The domain of* $f \circ g$ *is* $(-\infty, 2] \cup [2\infty)$*.*
The composite function $f \circ g$ *is continuous on its domain* $(-\infty, 2] \cup [2\infty)$*.*

Inverse of Continuous Function

Proposition

If f is an one-to-one continuous function, then its inverse f^{-1} is continuous.

So, we have more continuous functions: $\sin^{-1} x, \cos^{-1} x, \ldots$

Intermediate Value Theorem

For a function f which is continuous on a closed and bounded interval $[a, b]$, there are interesting and useful results. One of them is the Intermediate Value Theorem.

Intermediate Value Theorem

Theorem

Suppose that f is continuous on the closed and bounded interval $[a, b]$ where $f(a) \neq f(b)$.
Let p be any real number between $f(a)$ and $f(b)$.
Then there exists a number c in the open interval (a, b) such that $f(c) = p$.

Remark The number p is an **intermediate value** between $f(a)$ and $f(b)$.

Intermediate Value Theorem - Graphical Illustration

Intermediate Value Theorem: Applications

A typical use of the Intermediate Value Theorem is to locate a root of a function.

Recall that for a function f, we say $x = c$ is a root of f if $f(c) = 0$.

IVT: Application(1) Finding Roots

Suppose f is continuous on $[a, b]$ where $f(a)$ and $f(b)$ are opposite in signs. Then $p = 0$ is an intermediate value.
By the Intermediate Value Theorem, there is a real number $c \in (a, b)$ at which $f(c) = 0$.

Finding Roots

Example

For the polynomial $f(x) = x^3 - 5x^2 + 3$, do you think there is a root of f in the interval $(0, 1)$? Justify your answer.

Application(2) Intersection of Curves

Example

Use Intermediate Value Theorem to explain why the two curves $y = \cos x$ and $y = x^2$ intersect at some point with x- coordinate c where $c \in (0, \frac{\pi}{2})$.

Application(2) Intersection of Curves

Solution

Curves $y = \cos x$ *and* $y = x^2$ *intersect at some point with* x*- coordinate* c *means that*

$$\cos c = c^2.$$

By shifting all expressions to one side of the equation, the above equation is equivalent to

$$\cos c - c^2 = 0.$$

To find an appropriate function f, *the above suggests* $f(c) = \cos c - c^2$ *and* 0 *is the intermediate value.*
Thus, we consider

$$f(x) = \cos x - x^2 \text{ where } x \in [0, \pi/2].$$

Solution (cont'd)

Thus, we consider

$$f(x) = \cos x - x^2 \text{ where } x \in [0, \pi/2].$$

Note that f is continuous on $[0, \pi/2]$, the values $f(0) = 1 > 0$ and $f(\pi/2) = -(\pi/2)^2 < 0$.
By Intermediate Value Theorem, there is a real number $c \in (0, \pi/2)$ at which $f(c) = 0$, which means that the curves intersect at $x = c$.

Preserves Connectedness

The Intermediate Value Theorem says that a continuous function preserves 'connectedness'.

Corollary

Suppose f is continuous on (a, b) and f is not a constant function. Then the range of f is an interval.

Extreme Value Theorem

Another important result related to a continuous function defined on an closed and bounded interval $[a, b]$ is the Extreme Value Theorem, which will be stated without proof. To discuss this theorem, we have to know what is meant by extreme values.

Global Maximum/Minimum

Definition

(a) A function f has a global maximum (also known as an absolute maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the maximum value of f on D.

(b) Similarly, f has a global minimum (also known as an absolute minimum) at c if $f(c) \leq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the minimum value of f on D.

(c) The maximum and minimum values of f are called the extreme values of f.

Example: A non-continuous function

Example

Consider $f : [-1, 1] \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} -x & \text{if } x < 0; \\ 1 & \text{if } x \geq 0. \end{cases}$$

Does f have a global maximum and a global minimum on $[-1, 1]$?

Example: A non-continuous function

Solution

Note that f *is not continuous on* $[-1, 1]$.
For $x \in [-1, 1]$ *we have* $0 < f(x) \leq 1$. *The range of* f *is* $(0, 1]$.
The function f *has a global maximum since* $f(0) = 1$, *the maximum value for* f *on* $[-1, 1]$.
However, f *has no global minimum.*

Example

Example

Let

$$f(x) = \frac{1}{x}, x \in (0, 1]$$

Then f has no global (or local) maximum. Its global minimum is $1 = f(1)$.

Note that f is continuous on $(0, 1]$ but $(0, 1]$ is not a closed interval.

Extreme Value Theorem

The important theorem below says that global maximum and global minimum are guaranteed when both continuity and closed-and-bounded condition are satisfied. We shall state without proof the Extreme Value Theorem.

Theorem (Extreme Value Theorem)

If f is a continuous function on a closed and bounded interval $[a, b]$, then there are points c_1 and c_2 in $[a, b]$ such that $f(c_1) = m$ is a global minimum and $f(c_2) = M$ is a global maximum for f.

Extreme Value Theorem - Consequence

As a consequence of both Intermediate Value Theorem and Extreme Value Theorem, we have:

Corollary

If $f : [a, b] \to \mathbb{R}$ is a continuous function, then the range R_f of f is a singleton $\{f(a)\}$ or a closed and bounded interval $[f(c_1), f(c_2)]$, for some $c_1 \in [a, b]$ and $c_2 \in [a, b]$.

Example

Example

Determine the range of the function f defined by $f(x) = 3x^2, x \in [-2, 1]$.

Solution The function $f(x) = 3x^2$ is continuous on $[-2, 1]$.

For $-2 \leq x \leq 1$, note that $f(x) = 3x^2 \leq 3(-2)^2 = 12 = f(-2)$ and $f(x) = 3x^2 \geq 3(0)^2 = f(0)$. That is, $f(0) \leq f(x) \leq f(-2)$.

Thus, the global maximum of f on $[-2, 1]$ is $f(-2) = 12$ while its global minimum is $f(0) = 0$.

The range of f is $[0, 12]$.

Remark

In the topic on applications of differentiation, we shall learn a systematic way to search for extreme values of a continuous function over a closed and bounded interval $[a, b]$.