MH1810 Math 1 Part 2 Limits and Continuity

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Limit Laws for Infinite Limits

Theorem

Suppose $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} g(x) = \infty$, *and* $\lim_{x \to a} h(x) = c$, *where* c *is a constant. Then*

1. $\lim_{x \to a} (f(x) + g(x)) = \infty$
   $\lim_{x \to a} (f(x) + h(x)) = \infty.$
2. $\lim_{x \to a} f(x) \cdot g(x) = \infty$
3. $\lim_{x \to a} f(x) \cdot h(x) = \infty$ *if* $c > 0$
   $\lim_{x \to a} f(x) \cdot h(x) = -\infty$ *if* $c < 0.$
4. $\lim_{x \to a} \frac{1}{f(x)} = 0.$
   The same Laws holds for $\lim_{x \to a^+}$ *and* $\lim_{x \to a^-}$.

Example

Example

Evaluate $\lim_{x \to \pi/2^-} (\tan x + 2 \sin x)$

Solution

Note that $\lim_{x \to \pi/2^-} (\tan x) = +\infty$ *and* $\lim_{x \to \pi/2^-} (2 \sin x) = 2 \sin(\pi/2) = 2.$

Thus,

$$\lim_{x \to \pi/2^-} (\tan x + 2 \sin x) = +\infty.$$

Example

Example

Evaluate $\lim_{x\to\pi/2^-}(-3\tan x\sin x)$

Solution

Since $\lim_{x\to\pi/2^-}(\tan x) = +\infty$ *and* $\lim_{x\to\pi/2^-}(-3\sin x) = -3\sin(\pi/2) = -3$, *we have*

$$\lim_{x\to\pi/2^-} -3\tan x\sin x = -\infty.$$

Example

Example

Evaluate $\lim_{x \to \pi/2^+} \frac{x}{\tan x}$

Solution

Note that $\lim_{x \to \pi/2^+} x = \pi/2$ *and* $\lim_{x \to \pi/2^+} \tan x = -\infty$. *Thus, we have*

$$\lim_{x \to \pi/2^+} \frac{x}{\tan x} = 0.$$

Some Useful Techniques

When we have $\lim_{x \longrightarrow a} f(x) = 0$, intuitively, we know that $\lim_{x \longrightarrow a} \frac{1}{f(x)}$ will diverge.

One of the following will hold:

- $\lim_{x \longrightarrow a} \frac{1}{f(x)} = \infty$,

Question: How do we know which one will hold?

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Some Useful Techniques

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One of the following will hold:

- $\lim_{x \to a} \frac{1}{f(x)} = \infty$,
- $\lim_{x \to a} \frac{1}{f(x)} = -\infty$,
- $\lim_{x \to a} \frac{1}{f(x)}$ does not exist.

Question: How do we know which one will hold?

Some Useful Techniques

Theorem

Suppose $\lim_{x \longrightarrow a} f(x) = 0$.

(a) If $f(x) > 0$ *on some deleted neighborhood of* a*, then* $\lim_{x \longrightarrow a} \frac{1}{f(x)} = \infty$,

(b) If $f(x) < 0$ *on some deleted neighborhood of* a*, then* $\lim_{x \longrightarrow a} \frac{1}{f(x)} = -\infty$,

(c) Otherwise, $\lim_{x \longrightarrow a} \frac{1}{f(x)}$ *does not exist.*

Note: A neigborhood of a is an interval $(a - \delta, a + \delta)$ containing a.
A deleted neighborhood of a is the set $(a - \delta, a + \delta) - \{a\}$ ($=(a - \delta, a) \cup (a, a + \delta)$).

Some Useful Techniques - One sided limits

For one-sided limits, we expect similar results to hold. We only state for $\lim\limits_{x\to a^+}$. You can write down the result for $\lim\limits_{x\to a^-}$.

Proposition

Suppose $\lim\limits_{x\to a^+} f(x) = 0$.

(a) *If* $f(x) > 0$ *for* $x \in (a, a+\delta)$, *then* $\lim\limits_{x\to a^+} \dfrac{1}{f(x)} = \infty$.

(b) *If* $f(x) < 0$ *for* $x \in (a, a+\delta)$, *then* $\lim\limits_{x\to a^+} \dfrac{1}{f(x)} = -\infty$.

Example

Example

Evaluate $\lim\limits_{x \to 1^+} \frac{1}{1-x^3}$.

Solution

Note that $\lim\limits_{x \to 1^+} 1 - x^3 = 0$.
For $x > 1$, *note that* $x^3 > 1$ *and hence* $1 - x^3 < 0$.
Therefore, we conclude that $\lim\limits_{x \to 1^+} \frac{1}{1-x^3} = -\infty$.

Example

Example

Evaluate $\lim\limits_{x \to -2^-} \frac{x-1}{x+2}$

Solution

Note that $\lim\limits_{x \to -2^-} x - 1 = -3,$

and that $\lim\limits_{x \to -2^-} \frac{1}{x+2} = -\infty$, *since* $x + 2 < 0$ *for* $x < -2$.

Therefore, $\lim\limits_{x \to -2^-} \frac{x-1}{x+2} = \infty$.

Limits at Infinity

Limit Theorem or laws also hold for $\lim\limits_{x\to\infty}$ and $\lim\limits_{x\to-\infty}$ provided the respective limits exist.
For example, suppose $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} g(x) = M$ exist. Then we have

$$\lim_{x\to\infty} (f(x) + g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = L + M,$$

$$\lim_{x\to\infty} (f(x) \cdot g(x)) = \left(\lim_{x\to\infty} f(x)\right) \cdot \left(\lim_{x\to\infty} g(x)\right) = LM.$$

and

$$\lim_{x\to\infty} (f(x))^n = \left(\lim_{x\to\infty} f(x)\right)^n = L^n, n \in \mathbb{Z}^+.$$

Some Useful Limits

Theorem

(a) If n is a positive integer, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \textit{ and } \lim_{x \to -\infty} \frac{1}{x^n} = 0.$$

(b) If m and n are positive integers, then

$$\lim_{x \to \infty} \frac{1}{x^{m/n}} = 0 \textit{ and}$$

$$\lim_{x \to -\infty} \frac{1}{x^{m/n}} = 0,$$

provided n is an odd integer.

Examples

Example

(a) $\lim_{x \to \infty} \frac{1}{x^5} = 0.$

(b) $\lim_{x \to -\infty} \frac{1}{x^{7/3}} + e^x = 0.$

(c) $\lim_{x \to \infty} \frac{1}{x^{3/4}} = 0.$

Evaluating Limits at Infinity for Rational Functions

Example (Divide by highest power of x)

Evaluate $\lim\limits_{x\to\infty} \frac{x+4}{x^2-6x+5}$, if it exists.

Solution

$$\lim_{x\to\infty} \frac{x+4}{x^2-6x+5} = \lim_{x\to\infty} \frac{x+4}{x^2-6x+5} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{4}{x^2}}{1-\frac{6}{x}+\frac{5}{x^2}} = \frac{0+0}{1-0+0} = 0.$$

Example

Example

Evaluate $\lim\limits_{x\to\infty} \frac{x^3+4x-5}{7x^3-6x+5}$

Solution

$$\lim_{x\to\infty} \frac{x^3+4x-5}{7x^3-6x+5}$$

$$= \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{4x}{x^3}-\frac{5}{x^3}}{\frac{7x^3}{x^3}-\frac{6x}{x^3}+\frac{5}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{4}{x^2}-\frac{5}{x^3}}{7-\frac{6}{x^2}+\frac{5}{x^3}} = \frac{1}{7}$$

Limits of Hyperbolic Functions

Proposition

(a)

$$\lim_{x\to\infty} \sinh x = +\infty \text{ and } \lim_{x\to-\infty} \sinh x = -\infty.$$

(b)

$$\lim_{x\to\infty} \cosh x = +\infty \text{ and } \lim_{x\to-\infty} \cosh x = +\infty.$$

(c)

$$\lim_{x\to\infty} \tanh x = +1 \text{ and } \lim_{x\to-\infty} \tanh x = -1.$$

Limits of Hyperbolic Functions

Proof of (a):
Since $\lim_{x\to\infty} e^x = +\infty$ and $\lim_{x\to-\infty} e^x = 0$, we have

$$\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = +\infty,$$

and

$$\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{2} = -\infty.$$

(b) (Exercise.)

Limits of Hyperbolic Functions

Proof of (c)

$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Since $\lim_{x\to\infty} e^x = +\infty$ and $\lim_{x\to\infty} e^x = 0$, we divide both numerator and denominator by e^x to obtain the following

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{1/e^x}{1/e^x}$$

$$= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1.$$

(Exercise.) $\lim_{x\to-\infty} \tanh x = -1.$

Limits involving Square Roots

Example

Evaluate $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5}$.

Solution

For $x > 0$, note that $x = \sqrt{x^2}$.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2}}{3}$$

Limits involving Square Roots

Example

$\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5}$

Solution

For $x < 0$, *note that* $x = -\sqrt{x^2}$.

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\frac{1}{-\sqrt{x^2}}}{\frac{1}{x}} = \lim_{x \to -\infty} \frac{-\sqrt{\frac{2x^2}{x^2} + \frac{1}{x^2}}}{\frac{3x}{x} - \frac{5}{x}}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} = -\frac{\sqrt{2}}{3}.$$

Problem

Problem

If $\dfrac{4x-1}{x} < f(x) < \dfrac{4x^2+3x}{x^2}$, *find* $\lim\limits_{x \to \infty} f(x)$.