MH1810 Math 1 Part 2 Limits and Continuity

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Limit Theorems

Theorem

(a) $\lim_{x\to a} C = C$.
(b) $\lim_{x\to a} x = a$.

Examples

(a) $\lim_{x\to 4} 1965 = 1965$.
(b) $\lim_{x\to\sqrt{3}} x = \sqrt{3}$.

Limit Laws

The following properties hold, whenever the limits on the right hand side exist.

Theorem

(1) $\lim_{x\to a} Cf(x) = C\lim_{x\to a} f(x)$
(2) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

Example

(1) $\lim_{x\to\pi} 7x = 7\lim_{x\to\pi} x = 7\pi$
(2) $\lim_{x\to\pi}(7x+\sqrt{3}) = \lim_{x\to\pi} 7x + \lim_{x\to\pi}\sqrt{3} = 7\pi + \sqrt{3}$

Limit Laws

Theorem

(3) $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
(4) $\lim_{x\to a} (f(x))^n = (\lim_{x\to a} f(x))^n$*, where n is a positive integer.*

Example

(3) $\lim_{x\to\pi}(7x)(x) = \lim_{x\to\pi}(7x) \cdot \lim_{x\to\pi}(x) = (7\pi)(\pi) = 7\pi^2$.
(4) $\lim_{x\to a} (x)^n = (\lim_{x\to a} x)^n = x^n$, where n is a positive integer.

Limit Laws

Theorem

$$(5)\ \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)},\quad if\ \lim_{x\to a}g(x)\neq 0$$

Example

$$(5)\ \lim_{x\to\pi}\frac{7}{x^3}=\frac{\lim\limits_{x\to\pi}7}{\lim\limits_{x\to\pi}x^3}=\frac{7}{\pi^3},\quad since\ \lim_{x\to\pi}x^3=\pi^3\neq 0$$

NOTE All the above properties hold for

$$\lim_{x\to a^+}\quad \lim_{x\to a^-}\quad \lim_{x\to\infty}\quad \&\quad \lim_{x\to-\infty}.$$

Limit Laws

Example

Evaluate $\lim_{x\to\pi^-} x^3(2x-5)$.

Solution

As $\lim_{x\to\pi^-} x^3 = (\lim_{x\to\pi^-} x)^3 = \pi^3$ *and* $\lim\limits_{x\to\pi^-}(2x-5) = \lim\limits_{x\to\pi^-}(2x) - \lim\limits_{x\to\pi^-} 5 = 2\pi - 5$, *we have*

$$\lim_{x\to\pi^-} x^3(2x-5) = \left(\lim_{x\to\pi^-} x^3\right)\left(\lim_{x\to\pi^-}(2x-5)\right) = \pi^3(2\pi-5).$$

Remark We will not write out all these details from now, but you should understand that it is the limit theorems that allow us to evaluate limits of sums, products, quotients and other expressions.

Limits of a Polynomial

It follows from limits laws that

Theorem

For a polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} \cdots \ a_2 x^2 + a_1 x + a_0,$$

we have $\lim_{x \to a} p(x) = p(a)$.

We can simply **substitute** the value of a into the polynomial $p(x)$ to obtain the limit of $p(x)$ at a.

Example

$\lim_{x\to 1}\left(2x^5 - \pi x^3 + \frac{1}{9}x - \sqrt{5}\right) = 2 - \pi + \frac{1}{9} - \sqrt{5}.$

Rational Functions

A function $f(x)$ is a **rational function** if it is a quotient of two polynomials, i.e., $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials.

Theorem

If $f(x) = \dfrac{p(x)}{q(x)}$ *and* a *is such that* $q(a) \neq 0$, *then, using the quotient rule, we have*

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

I.e., we can also simply substitute the value of a into the rational function $f(x) = \dfrac{p(x)}{q(x)}$, to obtain the limit of $f(x)$ at a, provided $q(a) \neq 0$.

Rational Functions

Example

Evaluate $\lim\limits_{x \to 3} \dfrac{3x+6}{x^2-4}$.

Solution

$$\lim_{x \to 3} \frac{3x+6}{x^2-4}$$
$$= \frac{3(3)+6}{3^2-4} = \frac{15}{5} = 3.$$

Substitution May Fail

We cannot apply substitution to find the following limits

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 9},$$

because $\lim\limits_{x \to 3} x^2 - 9 = 0$.

WARNING In general, we cannot simply substitute values of a directly into the function $f(x)$ to obtain the limit of $f(x)$ at a. Such substitution holds when the functions involved is continuous at a. We shall discuss this concept of continuity in the next section.

Continuity

Most functions you have come across in your pre-university mathematics courses are nice functions in the sense that the curves of their respective graphs are 'continuous'.

Continuity

Such nice functions allow us to substitute the value c directly into $f(x)$ in the evaluation of $\lim_{x \longrightarrow c} f(x)$. A mathematical name is 'continuity at $x = c$'.

Continuity at a Point

Definition (Continuity at a Point)

Let f be a function defined on an interval I and let c be an interior point of I. We say that f is continuous at $x = c$ if

$$\lim_{x \longrightarrow c} f(x) = f(c).$$

In words, the definition tells us that

The function f is continuous at $x = c$ means that the limit $\lim\limits_{x \longrightarrow c} f(x)$ can be obtained by substituting $x = c$ into $f(x)$.

Basic Functions

Polynomials, rational functions, $\sqrt[n]{x}$, $\sin x$ $\cos x$, $\tan x$, e^x and $\ln x$ are continuous at every point at which the function is defined.

Limits of Basic Functions

Theorem

(a) $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, *where n is a positive integer.*
For an odd integer n, $c \in \mathbb{R}$. *For an even integer n, note that* $c > 0$.
(b) $\lim_{x \to c} \sin x = \sin c$
(c) $\lim_{x \to c} \cos x = \cos c$
(d) $\lim_{x \to c} \tan x = \tan c$, *wherever* $\tan c$ *is defined.*

Limits of Basic Functions

Theorem

(e) for any $b > 0$, $\lim_{x \to c} \; b^x = b^c$, *in particular,* $\lim_{x \to c} \; e^x = e^c$
(f) $\lim_{x \to c} \; \ln x = \ln c$, *where* $c > 0$
(g) $\lim_{x \to c} \; \sinh x = \sinh c$
(h) $\lim_{x \to c} \; \cosh x = \cosh c$
(i) $\lim_{x \to c} \; \tanh x = \tanh c$, *wherever* $\tan c$ *is defined.*
(j) Inverse trigonometric functions and inverse hyperbolic functions are continuous at c *which are not end-points.*

Test for Continuity

To check if a function f is continuous at a point $x = c$, we have to check that $\lim_{x \to c} f(x) = f(c)$. This means that

(i) $f(c)$ is defined.

(ii) the limit $\lim\limits_{x \to c} f(x)$ exists.

(iii) $f(c)$ and $\lim\limits_{x \to c} f(x)$ are equal.

Example

Example

Consider the function

$$f(x) = \frac{1-x^2}{1-x}.$$

Is f continuous at $x = 1$?

Solution

(i) $f(1)$ *is not defined.*

Thus f is not continuous at $x = 1$.

Note that (ii) $\lim_{x \to 1} f(x) = 2$.

Example

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{for } x = 2, \\ 0 & \text{for } x \neq 2. \end{cases}$$

Is f continuous at $x = 2$?

Solution

$$f(x) = \begin{cases} 1 & \text{for } x = 2, \\ 0 & \text{for } x \neq 2. \end{cases}$$

Note $f(2) = 1$.

- $\lim_{x \to 2} \ f(x) = \lim_{x \to 2} \ 0 = 0$.

Solution

$$f(x) = \begin{cases} 1 & \text{for } x = 2, \\ 0 & \text{for } x \neq 2. \end{cases}$$

Note $f(2) = 1$.

- $\lim_{x \to 2} \ f(x) = \lim_{x \to 2} \ 0 = 0$.
- So $\lim_{x \to 2} \ f(x) \neq f(2)$.

Solution

$$f(x) = \begin{cases} 1 & \text{for } x = 2, \\ 0 & \text{for } x \neq 2. \end{cases}$$

Note $f(2) = 1$.

- $\lim_{x \to 2} f(x) = \lim_{x \to 2} 0 = 0$.
- So $\lim_{x \to 2} f(x) \neq f(2)$.
- We conclude that f is not continuous at $x = 2$.

Example

Example

Let

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

Is f continuous at $x = 0$?

Solution

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

Note that $f(0) = 1$.

However, the limit $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \sin \dfrac{1}{x}$ does not exist, so f is not continuous at $x = 0$.

One Sided Continuity

Definition

(One sided continuity) We say that f is continuous from the left at $x = c$ if

$$\lim_{x \to c^-} f(x) = f(c)$$

and that f is continuous from the right at $x = c$ if

$$\lim_{x \to c^+} f(x) = f(c).$$

Example

Example

Discuss whether the Heaviside function H defined by

$$H(x) = \begin{cases} 0 & ; \text{ for } x < 0, \\ 1 & ; \text{ for } x \geq 0. \end{cases}$$

is continuous at $x = 0$.If it is not, is it continuous from the left or right of $x = 0$?

Solution

Note that $\lim\limits_{x \to 0^-} H(x) = 0$ and $\lim\limits_{x \to 0^+} H(x) = 1$.
Since $\lim\limits_{x \to 0^-} H(x) \neq \lim\limits_{x \to 0^+} H(x)$, the limit $\lim\limits_{x \to 0} H(x)$ does not exist.
Therefore, $H(x)$ is not continuous at $x = 0$.

Since $\lim\limits_{x \to 0^+} H(x) = 1 = H(0)$, H is continuous from the right at $x = 0$.

As $\lim\limits_{x \to 0^-} H(x) = 0 \neq H(0)$, H is not continuous from the left at $x = 0$.

Properties on Continuity

It follows readily from limit laws that
If f and g are continuous at $x = c$, then

- $f \pm g$,
- $(f \cdot g)$ and
- f/g (provided $g(c) \neq 0$)

are continuous at $x = c$.

This is because,

$$\lim_{x \to c} (f \pm g)(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

$$= f(c) \pm g(c) = (f \pm g)(c).$$

Example

Evaluate $\lim_{x\to 1}\left(x^{-2}-4x^{1/7}-5x^5+\sqrt{2}x+\pi\right)$.

[Solution] Note that $x^{-2}, 4x^{1/7}, 5x^5$, $\sqrt{2}x$ are continuous at $x=1$. Therefore, the function $\left(x^{-2}-4x^{1/7}-5x^5+\sqrt{2}x+\pi\right)$ is continuous at $x=1$, and we have

$$\lim_{x\to 1}\left(x^{-2}-4x^{1/7}-5x^5+\sqrt{2}x+\pi\right)$$

$$=1^{-2}-4(1^{1/7})-5(1^5)+\sqrt{2}(1)+\pi=-8+\sqrt{2}+\pi.$$

Limit Law for Composite Functions

The limit law for composition requires some additional conditions. We record one result here which requires the continuity of the outermost function.

Limit Law for Composite Functions

Theorem

Suppose $\lim_{x \to c} g(x) = b$. *If* $f(y)$ *is continuous at* $y = b$, *i.e.,* $\lim_{y \to b} f(y) = f(b)$, *then*

$$\lim_{x \to c} f(g(x)) = \lim_{y \to b} f(y).$$

That is,

$$\lim_{x \to c} f(g(x)) = \lim_{y \to b} f(y) = f(b) = f\left(\lim_{x \to c} g(x)\right).$$

The theorem says that we can interchange the order of taking limit and applying the function f.

Limit Law for Composite Functions

Corollary

1. $\lim\limits_{x \to c} \sqrt[n]{g(x)} = \sqrt[n]{\lim\limits_{x \to c} g(x)}$.
2. $\lim\limits_{x \to c} e^{g(x)} = e^{\lim_{x \to c} g(x)}$.

Examples

The following functions are continuous at the specified $x = c$. Hence, the limits can be evaluated by direct substitution.

(a) $\lim_{x \to 7} \sqrt[3]{x+20}\ \cos(\frac{\pi x}{2}) = \sqrt[3]{7+20}\ \cos(\frac{\pi(7)}{2}) = 3(0) = 0$

(b) $\lim_{x \to e^2} \frac{\sin(x - e^x)}{\ln x} = \frac{\sin(e^2 - e^{e^2})}{\ln(e^2)} = \frac{\sin(e^2 - e^{e^2})}{2}$.

Eliminating Zero Denominators

The next example demonstrates a special situation for rational functions when both numerator and denominator have a common factor $(x - a)$.

Example

Evaluate the limit $\lim_{x \to -2} \frac{3x+6}{x^2-4}$.

Question Can we apply limit law as follows:

$$\lim_{x \to -2} \frac{3x + 6}{x^2 - 4} = \frac{\lim_{x \to -2} 3x + 6}{\lim_{x \to -2} x^2 - 4}?$$

Eliminating Zero Denominators

Example

Evaluate the limit $\lim_{x \to -2} \frac{3x+6}{x^2-4}$.

[Solution] We should simplify first:

$$\lim_{x \to -2} \frac{3x+6}{x^2-4} = \lim_{x \to -2} \frac{3(x+2)}{(x+2)(x-2)}$$

$$= \lim_{x \to -2} \frac{3}{x-2} = \frac{3}{-2-2} = -\frac{3}{4}.$$

Eliminating Zero Denominators

Example

$$\lim_{x \to -3} \frac{x^3+4x^2+4x+3}{-x^3-2x^2+5x+6}$$

Since both polynomials have a zero at $x = -3$, the Factor Theorem tells us that both polynomials have a factor $(x + 3)$.

Eliminating Zero Denominators

By long division, we can find the other factors.

$$x^3 + 4x^2 + 4x + 3 = (x+3)(x^2 + x + 1),$$

$$-x^3 - 2x^2 + 5x + 6 = (x+3)(-x^2 + x + 2).$$

We evaluate the limit:

$$\lim_{x \to -3} \frac{x^3 + 4x^2 + 4x + 3}{-x^3 - 2x^2 + 5x + 6} = \lim_{x \to -3} \frac{(x+3)(x^2 + x + 1)}{(x+3)(-x^2 + x + 2)}$$

$$\lim_{x \to -3} \frac{x^2 + x + 1}{-x^2 + x + 2} = \frac{9 - 3 + 1}{-9 - 3 + 2} = -\frac{7}{10}.$$

We can make use of the simple factorization $a^2 - b^2 = (a-b)(a+b)$ in some expressions involving square root function.

Example

Find $\lim_{x \to 1^+} \frac{\sqrt{2-x}-1}{x-1}$.

Question: Can you evaluation the limit by substitution?

Example

Example

Find $\lim_{x \to 1^+} \frac{\sqrt{2-x}-1}{x-1}$.

Solution

$$\lim_{x \to 1^+} \frac{\sqrt{2-x}-1}{x-1} = \lim_{x \to 1^+} \frac{\sqrt{2-x}-1}{x-1} \frac{\sqrt{2-x}+1}{\sqrt{2-x}+1}$$

$$= \lim_{x \to 1^+} \frac{(2-x)-1}{(x-1)(\sqrt{2-x}+1)} = \lim_{x \to 1^+} \frac{-(x-1)}{(x-1)(\sqrt{2-x}+1)}$$

$$= \lim_{x \to 1} \frac{-1}{\sqrt{2-x}+1} = \frac{-1}{\sqrt{2-1}+1} = \frac{-1}{2}.$$

Squeeze Theorem

Theorem (Squeeze Theorem)

Suppose f, g and h are defined on an open interval I containing a, except possibly at $x = a$.
If $f(x) \leq g(x) \leq h(x)$ on I, except possibly at $x = a$, and $\lim_{x \longrightarrow a} f(x) = \lim_{x \longrightarrow a} h(x) = L$, then

$$\lim_{x \longrightarrow a} g(x) = L.$$

Example

Example

Evaluate $\lim\limits_{x \to 0} x^2 \sin(\frac{1}{x^2})$.

[COMMON WRONG SOLUTION]

$$\lim_{x \to 0} x^2 \sin(\tfrac{1}{x^2}) \overset{*}{=} \lim_{x \to 0} x^2 \lim_{x \to 0} \sin(\tfrac{1}{x^2}) = 0, \text{ since } \lim_{x \to 0} x^2 = 0.$$

The product rule for limits is wrongly applied in the above 'solution'. As $\lim\limits_{x \to 0} \sin(\dfrac{1}{x^2})$ does not exist, the product rule cannot be applied at (*).

A correct solution makes use of Squeeze Theorem.

Correct Solution

We have to construct two functions f and h with $g(x) = x^2 \sin(1/x^2)$.
Observe that for $x \neq 0$,

$$-1 \leq \sin(1/x^2) \leq 1.$$

Multiplying the above throughout by x^2, which is positive, gives

$$\underbrace{-x^2}_{f(x)} \leq \underbrace{x^2 \sin(1/x^2)}_{g(x)} \leq \underbrace{x^2}_{h(x)}.$$

Moreover, we have $\lim_{x \to 0} -x^2 = 0$ and $\lim_{x \to 0} x^2 = 0$.
By Squeeze Theorem, we conclude that $\lim_{x \to 0} x^2 \sin(1/x^2) = 0$.

One-sided Limits

Recall the following result:

Theorem (Equal One-sided Limits)

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

Example

Consider the function

$$f(x) = \begin{cases} -x & \text{if } x < -1, \\ x^2 & \text{if } |x| \leq 1, \text{ i.e., } -1 \leq x \leq 1, \\ 2 & \text{if } x > 1 \end{cases}$$

Determine whether each of the following limits exists. If it does, what is its value?

(a) $\lim\limits_{x \to -1^-} f(x)$

(b) $\lim\limits_{x \to -1^+} f(x)$

(c) $\lim\limits_{x \to -1} f(x)$

(d) $\lim\limits_{x \to 1} f(x)$

Solution

$$f(x) = \begin{cases} -x & \text{if } x < -1, \\ x^2 & \text{if } |x| \le 1, \text{ i.e., } -1 \le x \le 1, \\ 2 & \text{if } x > 1 \end{cases}$$

(a) $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-x) = -(-1) = 1$

(b) $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2) = (-1)^2 = 1$

(c) Since $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = 1$, the limit $\lim_{x \to -1} f(x)$ exists and $\lim_{x \to -1} f(x) = 1$.

Solution (d)

To determine whether $\lim_{x \to 1} f(x)$ exists, we evaluate $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$ as follows:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x)^2 = (1)^2 = 1$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} (2) = 2.$$

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, the limit $\lim_{x \to 1} f(x)$ does not exist.

Example

Example

Determine whether $\lim_{x \to 2} f(x)$ where

$$f(x) = \begin{cases} \frac{3x-6}{x^2-4} & \text{if } 0 < x < 2, \\ 0 & \text{if } x = 2, \\ \frac{x-2}{\sqrt{3-x}-1} & \text{if } 2 < x < 3. \end{cases}.$$