Algebra: Matrices II - Determinants

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Determinants of 2 x 2 Matrices

Recall that a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.

This special number $ad - bc$ is known as the determinant of the 2×2 square matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

It is denoted by the symbol $\det(A)$.

Determinants of n x n Matrices

For a general $n \times n$ square matrix A, where $n \geq 3$, we shall compute the $\det(A)$ inductively via Cofactor Expansion.

What are cofactors?

Cofactor of a Matrix

Definition

The (i, j)th cofactor of A, denoted by C_{ij} , is the product of number $(-1)^{i+j}$ and the determinant M_{ij} of the submatrix that remains after the ith-row and the jth-column are deleted from A. That is,

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Note: The number M_{ij} is called the (i, j)th minor of A.

Example

Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

(a) Find the $(1,1)$th cofactor and $(2,3)$th cofactor of A.
(b)Calculate C_{12} and C_{31}.

Matrix of Cofactor and Adjoint of a Matrix

Definition

Let A be an $n \times n$ matrix, and C_{ij} be its (i,j)th cofactor. Then the matrix C whose entries are cofactors:

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the matrix of cofactors from A. The transpose of this matrix is called the adjoint of A and is denoted by $\text{adj}(A)$.

Example

Example

Let $A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

(a) Obtain the cofactor matrix C of A and $\operatorname{adj}(A)$.

(b) What is $A \operatorname{adj}(A)$?

Example

(a) $C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$, $C_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} = 4$, $C_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 2 \\ 1 & 2 \end{vmatrix} = -8 \ldots$

So we have

$$C = \begin{bmatrix} 0 & 4 & -8 \\ -5 & 1 & 3 \\ 5 & -1 & 17 \end{bmatrix} \ \& \ \mathrm{adj}(A) = \begin{bmatrix} 0 & -5 & 5 \\ 4 & 1 & -1 \\ -8 & 3 & 17 \end{bmatrix}.$$

(b)

$$A\mathrm{adj}(A) = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20\ I.$$

Adjoint and Inverse

More generally, we have the following result for a general square matrix A, (from theory in linear algebra).

Theorem

$$A adj(A) = det(A) I.$$

In particular, If $det(A) \neq 0$, then A is invertible and we have

$$A\left(\frac{1}{det(A)} adj(A)\right) = I.$$

Hence we have a formula for the inverse A^{-1} as follows:

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

Determinants via Cofactors

Cofactors are used in the evaluation of determinants in an inductive way.

We begin with determinants of matrices of sizes 1×1 and 2×2.

The determinant of a 1×1 matrix $[a]$ is a, i.e., $\det([a]) = a$.

Determinants of 2 x 2 Matrices

The determinant of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. We write it as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \text{or} \quad \det(A) = ad - bc,$$

which can be obtained by computing the sum of the products on the rightward arrow and subtracting the products in the leftward arrow.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinants of n x n Matrices, n > 2

The determinant of an $n \times n$ matrix A can be found by summing the products of terms in the first row with the corresponding cofactors:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Example

Example

Find the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Example

Example

Find the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

- $C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$, $C_{12} = (-1)^{1+2} \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} = 4$, $C_{13} = (-1)^{1+3} \begin{vmatrix} -3 & 2 \\ 1 & 2 \end{vmatrix} = -8$

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- So,

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(0) + 5(4) + 0(-9) \\ &= 20 \end{aligned}$$

Determinants of n x n Matrices, n > 2

We may also calculate the determinant by cofactor expansion along other rows.

1. Suppose we perform cofactor expansion along the ith-row. Say the entries on the ith row are:

$$a_{i1} \quad a_{i2} \quad \cdots \quad a_{ik} \quad \cdots a_{in}$$

2. Multiply each entry a_{ik} with its corresponding cofactor C_{ik}, i.e., $a_{ik}C_{ik}$.

Add all the resulting products obtained in the last step gives the determinant of A, i.e.,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}.$$

Example

By cofactor expansion along the second row, find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

$$\det(A) = \begin{vmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= (-3)(-1)^{2+1}\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + (1)(-1)^{2+3}\begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix}$$

$$= (-3)(-5) + (2)(1) + (1)(3) = 20.$$

'Checker-board' matrix

- The 3 by 3 matrix S

$$S = \begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}.$$

- The 4 by 4 matrix S

$$S = \begin{bmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{bmatrix}.$$

Are called 'Checker-board' matrices.

Calculating Determinants via ‘Checker-board’ matrix

Using the ‘checker-board’ matrix, we may compute determinant of smaller size matrices easily. For example, the determinant of A, say along third row:

$$\det(A) = \begin{vmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= (1)\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} - (2)\begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix} + (1)\begin{vmatrix} 1 & 5 \\ -3 & 2 \end{vmatrix}$$

$$= 5 - 2 + 17 = 20$$

Determinant of Transpose

From theory of determinants, both matrices A and its transpose A^T have the same determinants.

$$\det(A) = \det(A^T).$$

Therefore, instead of performing cofactor expansion along a selected row, we may also evaluate the determinant of A by cofactor expansion along a selected column.

Determinant of via Cofactors along Column

1. Select a column of A, say jth column. (So, we say that we perform cofactor expansion along the jth-column.)
2. Multiply each entry a_{kj} of the selected row by its corresponding cofactor C_{kj}, i.e., $a_{kj}C_{kj}$.
3. Add all the resulting products obtained in the last step gives us the determinant of A, i.e.,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

Example

Example

Find the determinant using cofactor expansion along the second column.

$$\det(A) = \begin{vmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

Solution

$$= -(5)\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} + (2)\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - (2)\begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix}$$

$$= (-5)(-4) + (2)(1) - 2(1) = 20.$$

Determinant of 3 x 3 Matrices

The determinant of the 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is (cofactor expansion along first row)

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

Determinant of 3 x 3 Matrices

An easy way to help us in computing the determinant of 3×3 matrix (ONLY) is by recopying the first and second columns next to the third column of A and followed by computing the sum of the products on the rightward arrows and subtracting the products in the leftward arrows.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Example

(a) $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$

(b) $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix}$

Example

Find the determinant of the matrix B by cofactor expansion

$$B = \begin{bmatrix} 1 & 1 & 5 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Example

In general, one strategy for evaluating a determinant by cofactor expansion is to expand along a row or column having the largest number of zeros.

Some Useful Theorem

Theorem

(a) Suppose that A *is a square matrix with a row of zeros, then* $\det A = 0$.
(b) Suppose that A *has two rows (or columns) such that one is a multiple of the other, then* $\det A = 0$.
(c) Suppose that A *is a triangular matrix, then* $\det A =$ *product of the diagonal terms.*

Example

Example

The determinants of the following matrices are zero.

$$\begin{bmatrix} 4 & 9 & 8 & 5 \\ -5 & -2 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & \pi & \frac{1}{2} \end{bmatrix}, \qquad \begin{bmatrix} 4 & 9 & 8 & 5 \\ -5 & -2 & 0 & 7 \\ 6 & 8 & 2\pi & 1 \\ 3 & 4 & \pi & \frac{1}{2} \end{bmatrix}.$$

Example

(a) $\det((I_n) = 1$

(b) $$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 0 & -2 & 0 & 7 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{vmatrix} = 4(-2)(3)(\tfrac{1}{2}) = -12$$

(c) $$\begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix} = abcd$$

Determinant of Product

Theorem

For two $n \times n$ matrices A and B,

$$det(AB) = det(A)det(B).$$

Determinant and Invertibility

Theorem

Let A be an $n \times n$ square matrix. Then
A is invertible if and only if $\det(A) \neq 0$.
Moreover, if $\det(A) \neq 0$, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

(Equivalently, the matrix A is singular if and only if $\det(A) = 0$.)

($\Rightarrow$) If A is invertible then we have $AA^{-1} = I$.
Taking determinants of matrices on both sides, we have

$$\det(AA^{-1}) = \det(I),$$

$$\text{i.e. },\ \det(A)\det(A^{-1}) = 1.$$

Hence $\det(A) \neq 0$ and we have

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Determinant and Invertibility (Proof)

($\Leftarrow$) follows from

Theorem

$$Aadj(A) = det(A)I.$$

from Linear Algebra. (Proof omitted).

Cramer's Rule

For a linear system $\mathbf{Ax} = \mathbf{b}$ whose coefficient matrix A is invertible, there is a formula for its solution.

The formula is known as Cramer's rule. It is useful for studying the mathematical properties of a solution without the need for solving the system.

Linear Equations vs Non-linear Equations

- Linear equations:

$$x_1 - 2x_2 = 3$$

$$\sqrt{2}a + \frac{1}{3}b - 5c = 199$$

$$x + 6y - 10z + 4w = 8$$

- Non-linear equations:

$$\sqrt{x} + 6y - 10z + 4w = 8$$

$$xy + 6y - 10z + 4w = 8$$

$$x + 6\sin y - 10z + 4w = 8$$

System of Linear Equations

– a system of linear equation where there are a finite number of linear equations.

Example: 2 equations in 3 unknowns.

$$\left\{ \begin{array}{rcrcrcr} 7x_1 & - & 2x_2 & + & 5x_3 & = & 3 \\ 3x_1 & + & x_2 & - & 4x_3 & = & -2 \end{array} \right.$$

which can be expressed as a matrix equation:

$$\begin{pmatrix} 7 & -2 & 5 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

System of Linear Equations

The linear system

$$\begin{array}{rcrcrcr} 2u & - & v & + & w & = & 3 \\ u & + & v & - & 3w & = & 5 \\ 5u & - & 4v & + & 9w & = & 4 \end{array}$$

is equivalent to

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -3 \\ 5 & -4 & 9 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}.$$

Cramer's Rule

Theorem (Cramer's Rule)

If $\mathbf{Ax} = \mathbf{b}$ *is a system of n linear equations in n unknowns such that* $det(A) \neq 0$*, then the system has a unique solution, namely*

$$x_j = \frac{det(A_j)}{det(A)}, j = 1, 2, \ldots, n$$

where A_j *is the matrix*

Cramer's Rule

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} \cdots & a_{nn} \end{bmatrix},$$

the matrix obtained by replacing the entries in the jth column of $\mathbf{A}$ by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Example

Example

For each of the following linear systems, determine whether Cramer's Rule is applicable. If so, solve the linear system.
(a)

$$\begin{array}{rcrcl} 7x_1 & - & 2x_2 & = & 3 \\ 3x_1 & + & x_2 & = & 5 \end{array}$$

Example (a)

[Solution] Note that $A = \begin{pmatrix} 7 & -2 \\ 3 & 1 \end{pmatrix}$, and $\det(A) = 13 \neq 0$.
Thus, Cramer's Rule applies.

$$A_1 = \begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}, \det(A_1) = -13; x_1 = \frac{\det(A_1)}{\det(A)} = -1.$$

$$A_2 = \begin{pmatrix} 7 & 3 \\ 3 & 5 \end{pmatrix}, \det(A_2) = 26; x_2 = \frac{\det(A_2)}{\det(A)} = 2.$$

Example (b)

$$
\begin{aligned}
2a &+ 4b &= 3\\
3a &+ 6b &= 5
\end{aligned}
$$