MH1810 Math 1 Part 1 Algebra

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Imaginary number

Does the quadratic equation $x^2 + 1 = 0$ have a real root? That is, are there real numbers x at which $x^2 = -1$?

To deal with the above irreducible quadratic equation, a new symbol 'i' is introduced, where

$$i^2 = -1.$$

Thus, $x^2 + 1 = 0$ has two distinct roots namely i and $-i$.

Powers of i

$$i^2 = -1, \quad i^3 = (i^2)(i) = -i,$$

$$i^4 = (i^2)(i^2) = (-1)(-1) = 1, \quad i^5 = (i^4)(i) = i, \ldots$$

Let $k \in \mathbb{Z}$. Then we have

$$i^{4k} = (i^4)^k = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i.$$

Note: Values of i^n depends on the remainder when n is divided by 4.

Complex number

(i) A **complex number** z is a mathematical object of the form $x + iy$, where x, y are real numbers. We have $z = x + iy$.

(ii) The real numbers x and y are called the real part and imaginary part of the complex number z respectively. We denote the real and imaginary parts of a complex number z by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ respectively.

(ii) We represent the set of all complex numbers by $\mathbb{C}$.

Examples of complex numbers

$$3+5i, \quad 3.5-i, \quad -\sqrt{3}+i, \quad \pi+9i,$$

$$\mathrm{Re}(3+5i)=3 \text{ and } \mathrm{Im}(3+5i)=5.$$

Definition (Equality of complex numbers.)

Two complex numbers $z = x + iy$ and $z' = x' + iy'$, where x, x', y and y' are real numbers, are said to be **equal** if

$$x = x' \text{ and } y = y'.$$

That is, $\mathrm{Re}(z) = \mathrm{Re}(z')$ and $\mathrm{Im}(z) = \mathrm{Im}(z')$.

Example

Example

Suppose x and y are real numbers such that the two complex number $(2x-3)+5i$ and $(x+7)-(y+1)i$ are equal. Find the values of x and y.

- Comparing both real and imaginary parts of the complex number respectively, we obtain

$$(2x-3) = x+7, \quad \text{and} \quad 5 = -(y+1),$$

which gives $x = 10$ and $y = -6$.

Remark

- We may identify every real number $x \in \mathbb{R}$ as a complex number (why?). In view of this we may think of the set of real number as a subset of the set of complex numbers, i.e., $\mathbb{R} \subseteq \mathbb{C}$.

Remark

- We may identify every real number $x \in \mathbb{R}$ as a complex number (why?). In view of this we may think of the set of real number as a subset of the set of complex numbers, i.e., $\mathbb{R} \subseteq \mathbb{C}$.
- We say that a complex number $z = x + iy$ is **purely imaginary** if the real part of z, namely x, is zero.

Argand Diagram

The representation of the complex number $z = x + iy$ is said to be in **rectangular form**. By identifying each complex number $z = x + iy$ by the point with coordinate (x, y), we actually represent the complex number z by a unique point on the xy-plane.

Modulus of z

The modulus $|z|$ of the complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$

It is the distance of the point (x, y) from $(0, 0)$.

Argument of z

For $z = x + iy$, the angle where the line joining points $(0, 0)$ and (x, y) made with the positive x-axis is known an argument of z, denoted by $\arg(z)$.

The counter-clockwise direction is considered 'positive' direction, whereas the clockwise is considered 'negative' direction.

Argument of z

Therefore, $\arg(z)$ is the angle θ such that

$$x = |z| \cos\theta \;\&\; y = |z| \sin\theta.$$

Note that $\tan\theta = \frac{y}{x}$, if $x \neq 0$.
If $\arg(z) = \theta$ (radians), then

$$\arg(z) = \theta + 2k\pi \text{ for every integer } k.$$

In particular, when the angle θ is chosen such that $-\pi < \theta \leq \pi$, we say this is the **principal argument** of z. It is denoted by $\operatorname{Arg}(z)$.

Polar form of z

Using the modulus and argument we can express a complex number $z = x + iy$ as

$$z = r(\cos\theta + i\sin\theta),$$

where $r = |z|$ and θ is an argument of z.
This representation is known as the polar form (also known as trigonometric form) of z.
We also use the notation $\mathrm{cis}\theta$ for $(\cos\theta + i\sin\theta)$, and write $z = r\mathrm{cis}\theta$

Exponential form of z

The **exponential form** of a complex number $z = r(\cos\theta + i\sin\theta)$ is written as

$$re^{i\theta}.$$

- commonly used in electronics, engineering and physics;
- convenient in discussing multiplication, division of complex numbers;
- formally discussed in advanced courses in mathematics via series.

Example

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Let $z = 3 - 3i$.

(a) Find the modulus and principal argument of z, and hence find its polar representation.

(b) Write down the exponential form of z.

- First, find the modulus of z :

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- Note that $r = |z| = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$.

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- Thus, $\alpha = \frac{\pi}{4}$.
- We have $\arg(3 - 3i) = -\frac{\pi}{4}$.
- Thus the polar form of $z = 3 - 3i$ is

$$3\sqrt{2}\left(\cos(\frac{-\pi}{4}) + \sin(\frac{-\pi}{4})\right) = 3\sqrt{2}\,\mathrm{cis}(\frac{-\pi}{4}).$$

Example

We have

$$r = |z| = 3\sqrt{2}, \quad \arg(3 - 3i) = -\frac{\pi}{4}.$$

Thus the exponential form of $3 - 3i$ is

$$3\sqrt{2}e^{\frac{-\pi}{4}i}.$$

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- The principle argument is $\operatorname{Arg}(z) = \frac{\pi}{3}$.
- Therefore, we have
- $z = 5e^{\frac{-5\pi}{3}i} = 5e^{\frac{\pi}{3}i}$
- $= 5\underbrace{\cos\frac{\pi}{3}}_{1/2} + i\underbrace{\sin\frac{\pi}{3}}_{\sqrt{3}/2} = \frac{5}{2} + \frac{5\sqrt{3}}{2}i.$

Conjugate of a complex number

Definition

The **conjugate** of a complex number $z = x + iy$ is the complex number $\bar{z} = x - iy$.

Notation for the complex conjugate of z: $\bar{z}$ or z^*.
Argand Diagram representing z and $\bar{z}$:

Examples

z	$\bar{z}$ or z^*
$3+5i$	
10	
$3.5-i$	
$-\sqrt{3}+i$	
	$\pi+9i$
$-\sqrt{7}i$	

Conjugate in Polar Form

The conjugate of the complex number $z = r\,(\cos\theta + i\sin\theta)$ (in polar form) or $z = re^{i\theta}$ (in exponential form), is respectively

$$
\begin{aligned}
z^* &= r\,(\cos(-\theta) + i\sin(-\theta))\,, \text{ or} \\
z^* &= re^{-i\theta}.
\end{aligned}
$$

Theorem

Let $z = x + yi$, where x and y are real numbers.
(a) $(z^*)^* = \overline{(\bar{z})} = z$.
(b) z *is real if and only if* $z = \bar{z}$.
(c) z *is imaginary if and only if* $z = -\bar{z}$.
(d) $|z^*| = |z|$ *and* $\arg(z^*) = -\arg(z)$.

Addition and subtraction

Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we define

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2).$$

Example

(a) $(3 + 5i) + (3.5 - i) = 6.5 + 4i$

(b) $(-\sqrt{3} + i) - (\pi + 9i) = (-\sqrt{3} - \pi) + (-8)i$

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- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$.

To multiply two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we can perform the multiplication treating i as a symbol. But we replace i^2 by (-1) when we simplify it :

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + x_1 iy_2 + (iy_1)x_2 + (iy_1)(iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1) \end{aligned}$$

Example

$(3 + 5i) \cdot (2 - i) = 3(2) + (5i)(2) + (3)(-i) + (5i)(-i)$
$= 6 + 10i - 3i - (5i^2) = 11 + 7i.$

Algebraic Properties

Theorem

(i) $z \cdot 1 = z = 1 \cdot z$.

(ii) $z_1 \cdot z_2 = z_2 \cdot z_1$.

(iii) $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$.

(iv) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

(v) $z \cdot \overline{z} = |z|^2$. *In particular, if* $z \neq 0$, *then* $z \cdot \overline{z} > 0$.

(vi) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Product in Polar Form

Given two complex numbers expressed in polar form:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \;\&\; z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Their product is

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- $z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$
- $= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + (\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2) i\right]$

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- $= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right).$

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- $z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$
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- $= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right).$
- Then $|z_1 z_2| =$
- and $\arg(z_1 z_2) =$

Important Result

Theorem

(a) Modulus of product is the product of moduli:

$$|z_1 \cdot z_2| = r_1 r_2 = |z_1|\ |z_2|$$

(b) Argument of the product is the **sum** *of arguments:*

$$\arg(z_1 \cdot z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2).$$

This implies the complex number $z_1 \cdot z_2$ lies on the line obtained by rotating the line segment representing z_1 by the angle $\arg(z_2)$.

Represent the product on an Argand diagram:

Represent the product on an Argand diagram

In particular, for a complex number z, the complex number $z \cdot e^{i\theta}$ is represented on the Argand diagram by by rotating z through θ.

Division of complex numbers

Recall that to express $\dfrac{1}{3+2\sqrt{5}}$ in the form $a+b\sqrt{5}$, we use the conjugate $3-2\sqrt{5}$ of $3+2\sqrt{5}$ to perform the following

$$\frac{1}{3+2\sqrt{5}}\cdot\frac{3-2\sqrt{5}}{3-2\sqrt{5}}=\frac{3-2\sqrt{5}}{3^2+(2\sqrt{5})^2}=\frac{3}{29}-\frac{2}{29}\sqrt{5}.$$

Division of complex numbers

To divide a complex number $z_1 = x_1 + y_1 i$ by a non-zero complex number $z_2 = x_2 + y_2 i$ (i.e., $z_2 \neq 0$), we use the conjugate $\overline{z_2} = x_2 - y_2 i$ as follows:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \frac{z_1 \overline{z_2}}{|z_2|^2}$$

Note that $z_2 \cdot \overline{z_2} = x_2^2 + y_2^2$ is a positive real number.

Example

Express $\frac{3+5i}{2-i}$ in the form $a+bi$.

The conjugate of the denominator $2-i$ is $2+i$. We have

$$\frac{3+5i}{2-i} = \frac{(3+5i)(2+i)}{(2-i)(2+i)} = \frac{(6-5)+(10+3)i}{5} = \frac{1}{5} + \frac{13}{5}i.$$

Division in polar form

In polar form, we have $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, such that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$$

Thus, we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right).$$

Division in polar (exponential) form

Using the exponential form, for $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ we have

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}, \text{ and}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1 - \theta_2)} \text{ where } z_2 \neq 0.$$

Both coincide with the law of exponents we are familiar with in real numbers.

Example

Let $z = \cos\theta + i\sin\theta$. Find $|z|$ and show that $\frac{1}{z} = \bar{z}$.

Note that $|z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. Thus, we have $z\bar{z} = 1$ and

$$\frac{1}{z} = \frac{1}{z}\,\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \bar{z} = \cos\theta - i\sin\theta.$$

The Fundamental Theorem of Algebra

Theorem (The Fundamental Theorem of Algebra)

Every polynomial equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

in which the coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ *are any complex numbers, whose degree* n *is greater than or equal to one, and whose leading coefficient* a_n *is not zero, has exactly* n *roots in the complex number system, provided each multiple root of multiplicity* m *is counted as* m *roots.*

Proof (Omitted): Textbook on theory of complex analysis.

Solving Quadratic Equations

Consider a quadratic equation $ax^2 + bx + c = 0$, where a, b and c are real numbers.
Recall that its **discriminant** D, is defined as $D = b^2 - 4ac$.

(i) If $D > 0$, the quadratic equation $ax^2 + bx + c = 0$ has two distinct real roots given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(ii) If $D = 0$, the quadratic equation $ax^2 + bx + c = 0$ has repeat real roots given by

$$x = \frac{-b}{2a}.$$

(iii) If $D < 0$, the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots given by

$$x = \frac{-b \pm i\sqrt{-(b^2 - 4ac)}}{2a}.$$

Note that the two complex roots are conjugate of each other. When $D < 0$, the quadratic equation or expression is said to be *irreducible*.

Example

Example

Solve the quadratic equation $2x^2 - 3x + 5 = 0$

For the given quadratic equation $2x^2 - 3x + 5 = 0$, its discriminant D is $D = (-3)^2 - 4(2)(5) = -31 < 0$.
Thus, $2x^2 - 3x + 5 = 0$ is irreducible.

The two distinct roots are $\dfrac{3+\sqrt{31}i}{4}$ and $\dfrac{3-\sqrt{31}i}{4}$ which form a conjugate pair.

Question: From the from the above example, we see that the roots of the equation appear in conjugate pairs. Is this true in general?

Polynomial with Real Coefficients

Theorem

Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ *is a polynomial in* x *with real coefficients* a_k*'s. If* z *is a solution to* $p(x) = 0$, *then the conjugate* $\bar{z}$ *is also a solution of* $p(x) = 0$.

For example: suppose z_0 is a complex root of $9x^5 + 7x^2 - 6x + \pi = 0$, then $\bar{z}_0$ is also a complex root of $9x^5 + 7x^2 - 6x + \pi = 0$. Therefore, $(x - z_0)(x - \bar{z}_0)$ is a quadratic factor of $9x^5 + 7x^2 - 6x + \pi$. Moreover, $(x - z_0)(x - \bar{z}_0) = x^2 - (z_0 + \bar{z}_0)x + z\bar{z}_0$ is a real coefficient quadratic factor.

As a consequence of the Fundamental Theorem of Algebra and the above result, we have the following useful result.

Theorem

Every odd degree polynomial $p(x)$ with real coefficients has at least one real root.

For example: $9x^5 + 7x^2 - 6x + \pi = 0$ has at least one real root.

Example

Example

Let $z = (\cos\theta + i\sin\theta)$. Find expressions for z^2 and z^3 in the same form.

De Moivre's Theorem

Theorem (De Moivre's Theorem)

For every rational number n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

The Euler representation DeMoivre's Theorem is

$$\left(e^{i\theta}\right)^n = e^{i(n\theta)}.$$

Examples

(a) $(\cos\theta + i\sin\theta)^9$

(b) $(\cos\theta + i\sin\theta)^{-4}$

Example. Simplify each of the following complex numbers

(a) $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{-2}$

(b) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{9}$

Example

Express each of the following complex numbers in the form $(\cos\theta + i\sin\theta)^n$

(a) $\cos 7\theta + i\sin 7\theta$.

(b) $\cos 5\theta - i\sin 5\theta$.

PROOF of De Moivre's Theorem

We prove the theorem by considering two cases:

First Case: n is a non-negative integer, i.e., $n \geq 0$.

Second Case: n is a negative integer, i.e., $n < 0$.

Case: n is non-negative integer
We shall prove

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

for $n = 0, 1, 2, 3, \ldots$ by Mathematical Induction.

PROOF: n is an non-negative integer

1. Verify the result holds for $n = 0$

$$(\cos\theta + i\sin\theta)^0 = 1, \cos 0\theta + i\sin 0\theta = \cos 0 = 1.$$

2. Assume the result hold for some non-negative integer k

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$$

3. We shall prove the result holds for $k + 1$ i.e.,

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta.$$

Indeed:

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^{k}(\cos\theta + i\sin\theta)$$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

$$= \cos(k+1)\theta + i\sin(kn+1)\theta.$$

Therefore by Mathematical induction, $(\cos\theta + i\sin\theta)^{n} = \cos n\theta + i\sin n\theta$ for all **non-negative integer** n.

PROOF: n is a negative integer

Case n is a negative integer, i.e., $n = -1, -2, -3, \ldots$.

Let $n = -m$ where m is a positive integer. Note that

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \cdot \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2(m\theta) + \sin^2(m\theta)} = \cos m\theta - i\sin m\theta$$

$$= \cos(-m\theta) + i\sin(-m\theta) = \cos n\theta + i\sin n\theta.$$

We begin with an example to have a geometrical idea of finding roots of a complex number before we state the formula for all distinct nth roots of $z = r(\cos\theta + i\sin\theta)$.

Example

Find all distinct cube roots of $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

Distinct nth roots

Theorem (Distinct nth roots)

Consider a complex number z in polar form

$$z = r(\cos\theta + i\sin\theta), \ \textit{where } r > 0 \textit{ and } -\pi < \theta \le \pi.$$

Then the distinct nth roots of the complex number $z = r(\cos\theta + i\sin\theta)$ are

$$z_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), k = 0, 1, 2, 3, \ldots, n-1.$$

Distinct nth roots - exponential form

In exponential form, we have all n distinct nth roots f the complex number $z = re^{i\alpha}$ are

$$z_k = \sqrt[n]{r}\left(e^{i\,\frac{\theta+2k\pi}{n}}\right), k = 0, 1, 2, 3, \ldots, n-1.$$

The n integers can be chosen to be any n consecutive integers.

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Find all distinct 5^{th} roots of $\sqrt{3}+i$.

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- $z_2=2^{1/5}e^{\frac{25\pi}{30}i}$, $z_3=2^{1/5}e^{\frac{37\pi}{30}i}=2^{1/5}e^{\frac{-23\pi}{30}i}$, $z_4=2^{1/5}e^{\frac{49\pi}{30}i}=2^{1/5}e^{\frac{-11\pi}{30}i}$.

Distinct nth roots

Corollary

The n distinct nth roots of $\cos\theta + i\sin\theta$ are

$$w_k = cis\left(\frac{\theta+2k\pi}{n}\right) = \cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right),$$

$$k = 0, 1, 2, \ldots, n-1.$$

In exponential form, we have

$$w_k = e^{i\left(\frac{\theta+2k\pi}{n}\right)}, k = 0, 1, 2, \ldots, n-1.$$

Roots of Unity

Note that $1 = 1 + 0i = \cos 0 + i \sin 0 = \cos 2k\pi + i \sin 2k\pi$, where k is an integer. We call n-th roots of 1 the n-th roots of unity.

Corollary (nth roots of unity)

The n distinct nth roots of unity are

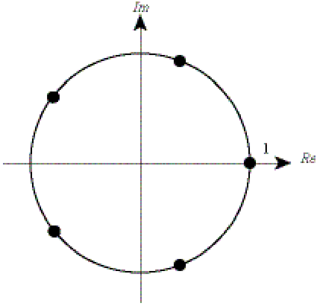
$$z_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, k = 0, 1, 2, 3, \ldots, n-1.$$

By De Moivre's Theorem, we have

$$z_k = (z_1)^k, \textit{where } z_1 = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

Roots of Unity

On the Argand diagram, all n-th roots of 1 are represented by points on the unit circle and they are equally spaced by $\frac{2\pi}{n}$:



Deriving Certain Trigonometric Identities I

Express $\cos n\theta$, $\sin n\theta$ and $\tan n\theta$ in terms of powers of $\cos\theta$, $\sin\theta$ and $\tan\theta$.
Tools:

$$\cos n\theta = \mathrm{Re}(\cos n\theta + i\sin n\theta) = \mathrm{Re}\left(\cos\theta + i\sin\theta\right)^n,$$

$$\sin n\theta = \mathrm{Re}(\cos n\theta + i\sin n\theta) = \mathrm{Im}\left(\cos\theta + i\sin\theta\right)^n,$$

Apply binomial expansion to $(\cos\theta + i\sin\theta)^n$
Notation used: $c \equiv \cos\theta$, $s \equiv \sin\theta$, $t \equiv \tan\theta$.

Example

Example

Express $\sin 3\theta$ in terms of powers of $\sin\theta$.

The first step is to note that

$$\sin 3\theta = \mathrm{Im}(\cos 3\theta + i\sin 3\theta)$$

Now, we apply de Moivre's theorem

$$\begin{aligned}
\sin 3\theta &= \mathrm{Im}(\cos 3\theta + i\sin 3\theta) \\
&= \mathrm{Im}\left(\cos\theta + i\sin\theta\right)^3 \quad \text{(why?)} \\
&= \mathrm{Im}\left(c + is\right)^3 \\
&= \mathrm{Im}\left(c^3 + 3c^2 is + 3ci^2 s^2 + i^3 s^3\right) \\
&= \mathrm{Im}\left(c^3 - 3cs^2 + i(3c^2 s - s^3)\right) \\
&= 3c^2 s - s^3
\end{aligned}$$

Example

Using $c^2 + s^2 = 1$, we have

$$\begin{aligned}
\sin 3\theta &= 3c^2 s - s^3 \\
&= 3(1 - s^2)s - s^3 \\
&= 3s - 4s^3 \\
&= 3\sin\theta - 4\sin^3\theta.
\end{aligned}$$

Example

From the above, we have also obtained an expression for $\cos 3\theta$:

$$\cos 3\theta = c^3 - 3cs^2 = c^3 - 3c(1 - c^2) = 4c^3 - 3c$$

Using the expression for both $\sin 3\theta$ and $\cos 3\theta$, we obtain a similar expression for $\tan 3\theta$:

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3c^2 s - s^3}{c^3 - 3cs^2}$$

$$= \frac{3c^2 s - s^3}{c^3 - 3cs^2} . \left(\frac{1/c^3}{1/c^3} \right) = \frac{3t - t^3}{1 - 3t^2}$$

Deriving Certain Trigonometric Identities II

Express $\cos^n\theta$ or $\sin^n\theta$ in terms of cosines and sines of multiples of θ, i.e. $\cos k\theta$, $\sin k\theta$.

Main Tool: Let $z = \cos\theta + i\sin\theta$, we have $\frac{1}{z} = \cos\theta - i\sin\theta$.

Thus we have $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$.

Deriving Certain Trigonometric Identities II

Next, we apply binomial expansion and group z^k and $\frac{1}{z^k}$ together.
By De Moivre's Theorem, we have

$$z^k = \cos k\theta + i \sin k\theta \text{ and } \frac{1}{z^k} = \cos k\theta - i \sin k\theta$$

which gives

$$z^k + \frac{1}{z^k} = 2\cos k\theta \text{ and } z^k - \frac{1}{z^k} = 2i \sin k\theta.$$

Thus, we obtain an expression involving sines and cosines of multiple of θ.

Example

Example

Prove that $\cos^3\theta = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)$

Proof.

Let $z = \cos\theta + i\sin\theta$. We have

$$\cos^3\theta = (\cos\theta)^3 = \left(\frac{1}{2}(z + \frac{1}{z})\right)^3$$

□

$$= \frac{1}{8}\left(z^3 + 3z^2\frac{1}{z} + 3z(\frac{1}{z})^2 + (\frac{1}{z})^3\right) = \frac{1}{8}\left((z^3 + \frac{1}{z^3}) + 3(z + \frac{1}{z})\right)$$

$$= \frac{1}{8}\left[2\cos 3\theta + 3(2\cos\theta)\right] = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right).$$