BBMerlion

Math A Level

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QUIZ 3 SOLUTION

Integral and McLaurin Series

Differential Equation

Permutation and Combination

Question 1

a. Let $u = \cos x$, $du = -\sin x \; dx$, $-du = \sin x\, dx$

$$\int 2(\cos x)^{-\frac{1}{2}} \sin x\, dx = \int 2u^{-\frac{1}{2}}(-du) = -2\int u^{-\frac{1}{2}}\, du = -2\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= -4u^{\frac{1}{2}} + C = -4(\cos x)^{\frac{1}{2}} + C$$

b. Let $u = \ln x$, $du = \frac{1}{x} dx$

$$\int \frac{(\ln x)^{-3}}{x} dx = \int u^{-3}\, du = -\frac{1}{2}u^{-2} + C$$

$$= -\frac{1}{2}(\ln x)^{-2} + C$$

Question 1

c. $\int_1^8 \frac{\log_4 \theta}{\theta} d\theta = \int_1^8 \frac{\ln \theta}{(\ln 4)(\theta)} d\theta$

Let $u = \ln \theta$, $du = \frac{1}{\theta} d\theta$

$$a = \ln 1 = 0, b = \ln 8$$

$$\int_1^8 \frac{\ln \theta}{(\ln 4)(\theta)} d\theta = \int_0^{\ln 8} \frac{u}{\ln 4} du = \frac{1}{2 \ln 4} u^2 \Big|_0^{\ln 8}$$

$$= \frac{(\ln 8)^2}{2 \ln 4}$$

d. Let $u = \sec \theta$, $du = \sec \theta \tan \theta \, d\theta$

$$a = \sec 0 = 1, b = \sec \frac{\pi}{3} = 2$$

$$\int_0^{\frac{\pi}{3}} \frac{\tan \theta}{\sqrt{2 \sec \theta}} d\theta = \int_0^{\frac{\pi}{3}} \frac{\sec \theta \tan \theta}{\sqrt{2}(\sec \theta)^{\frac{3}{2}}} d\theta = \int_1^2 \frac{1}{\sqrt{2} u^{\frac{3}{2}}} du$$

$$= \frac{1}{\sqrt{2}} \left[\frac{u^{-\frac{1}{2}}}{(-\frac{1}{2})} \right]_1^2 = \left[-\frac{2}{\sqrt{2u}} \right]_1^2 = \sqrt{2} - 1$$

Question 2

a. Area of cross-section:

$$A(x) = \frac{1}{2}(2\sqrt{x} - x)^2 \sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{4}(4x - 4x\sqrt{x} + x^2)$$

We set the integration limit to be a=0 and b=4, because that point is when the boundary equations intersect.

$$V = \frac{\sqrt{3}}{4}\int_0^4 (4x - 4x^{\frac{3}{2}} + x^2)\,dx$$

$$= \frac{\sqrt{3}}{4}\left[2x^2 - \frac{8}{5}x^{\frac{5}{2}} + \frac{x^3}{3}\right]_0^4 = \frac{\sqrt{3}}{4}\left(32 - \frac{8(32)}{5} + \frac{64}{3}\right)$$

$$= \frac{8\sqrt{3}}{15}$$

Question 2

b. We rotate the region enclosed by the curve $y = \sqrt{12\left(1 - \frac{4x^2}{121}\right)}$ and the x-axis around the x-axis. To find the volume we use disk method:

$$V = \int_a^b \pi R^2(x)\,dx = \int_{-\frac{11}{2}}^{\frac{11}{2}} \pi\left(12\left(1 - \frac{4x^2}{121}\right)\right)dx$$

$$= 12\pi\left[x - \frac{4x^3}{363}\right]_{-\frac{11}{2}}^{\frac{11}{2}} = 24\pi\left[\frac{11}{2} - \left(\frac{4}{363}\right)\left(\frac{11}{2}\right)^3\right]$$

$$= 88\pi$$

Question 2

c. $f(x) = \frac{x}{\sqrt{1-x^2}}, f(0) = 0$

$$f^{(1)}(x) = \frac{\sqrt{1-x^2} - x\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2} = \frac{1}{(1-x^2)^{\frac{3}{2}}}, f(0) = 1$$

$$f^{(2)}(x) = -\frac{3}{2}(1-x^2)^{-\frac{5}{2}}(-2x), f(0) = 0$$

$$f^{(3)}(x)$$
$$= -\frac{3}{2}(1-x^2)^{-\frac{5}{2}}(-2) + \frac{3}{2}\left(\frac{5}{2}\right)(1-x^2)^{-\frac{7}{2}}(-2x)(-2x), f(0) = 3$$

Question 3

a. i. $\frac{dy}{dt} + \frac{2y}{t} = t^2$

$$P(t) = \frac{2}{t}, Q(t) = t^2$$

$$\int P(t)\,dt = 2\ln|t|$$

$$v(t) = e^{\ln t^2} = t^2$$

$$y = \frac{1}{t^2}\int t^2 t^2\,dt = \frac{1}{t^2}\left(\frac{t^5}{5} + C\right)$$

From initial condition,

$$y(2) = 1 = \frac{8}{5} + \frac{C}{4}, C = -\frac{12}{5}$$

$$y = \frac{t^3}{5} - \frac{12}{5t^2}$$

Question 3

ii. $(t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2}$

$$\frac{ds}{dt} + \frac{2}{t+1}s = 3 + \frac{1}{(t+1)^3}$$

$P(t) = \frac{2}{t+1}$, $Q(t) = 3 + (t+1)^{-3}$

$$\int P(t)\,dt = \int \frac{2}{t+1}\,dt = 2\ln|t+1| = \ln(t+1)^2$$

$$v(t) = e^{\ln(t+1)^2} = (t+1)^2$$

$$s = \frac{1}{(t+1)^2}\int (t+1)^2(3 + (t+1)^{-3})\,dt$$

$$= \frac{1}{(t+1)^2}\int (3(t+1)^2 + (t+1)^{-1})\,dt$$

$$= \frac{1}{(t+1)^2}\left((t+1)^3 + \ln|t+1| + C\right), t > -1$$

Question 2

b. i. $\frac{di}{dt} + \frac{R}{L} i = 0$

$$\frac{1}{i} di = -\frac{R}{L} dt$$

$$\ln i = -\frac{Rt}{L} + C_1$$

$$i = Ce^{-\frac{Rt}{L}}$$

$$i(0) = I, C = I$$

$$i = Ie^{-\frac{Rt}{L}}$$

ii. $\frac{1}{2} I = Ie^{-\frac{Rt}{L}}$

$$e^{-\frac{Rt}{L}} = \frac{1}{2}, t = \frac{L}{R} \ln 2$$

iii. For $t = \frac{L}{R}, I = Ie^{-1}$

Question 4

a. Repetitions not allowed:

$$n = 5! = 120$$

Repetitions allowed:

$$n = 5^5 = 3125$$

b. To ensure there is no two ladies sit together, gentlemen and ladies have to sit alternately.

No. of arrangements of gentlemen = No. of arrangements of ladies =

$$n = 5! = 120$$

Total possible arrangements:

$$n_{total} = 120 \times 120 \times 2 = 28800$$

Question 4

c.

$$c = \frac{8!}{3!\,5!} \times \frac{7!}{4!\,3!} = 1960$$

If Miss X refuses to serve when Mr. Y is inside the committee,

$$c = (Mr.Y\ not\ inside\ committee) + (Mr.Y\ inside\ committee)$$

$$c = \frac{8!}{3!\,5!} \times \frac{6!}{4!\,2!} + \frac{7!}{3!\,4!} \times \frac{6!}{3!\,3!} = 1540$$

References

Thomas Calculus Early Transcedentals 12th Edition