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Math A level

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QUIZ 2 SOLUTION

Vectors

Complex Number

Differentiation

Question 1

a. Let $y = \frac{x}{1+x^2}$

Then,

$$\lim_{x \to -\infty} y = 0$$

$$\lim_{x \to \infty} y = 0$$

We need to find the maximum and minimum points of y, i.e.

$$\frac{dy}{dx} = \frac{\left(1 + x^2\right) - 2x^2}{(1 + x^2)^2} = 0$$

Therefore,

$$\left(1 + x^2\right) - 2x^2 = 0$$

$$1 - x^2 = 0$$

$$x = \pm 1$$

Question 1

Substituting x into y, we get:

For x = 1, $y = \frac{1}{1+1} = \frac{1}{2}$ (Max)

For x = -1 $y = -\frac{1}{1+1} = -\frac{1}{2}$ (Min)

Therefore, it is proven that

$$-\frac{1}{2} \leq y \leq \frac{1}{2}$$

for every value of x

Question 1

b. We can find the gradient of the cables by differentiating the circle because the cables are tangent to the circle.

By using implicit differentiation,

$$x^2 + y^2 = 225$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

For the left tangent:

$$\frac{dy}{dx} = \frac{12}{-9} = -\frac{4}{3}$$

Question 1

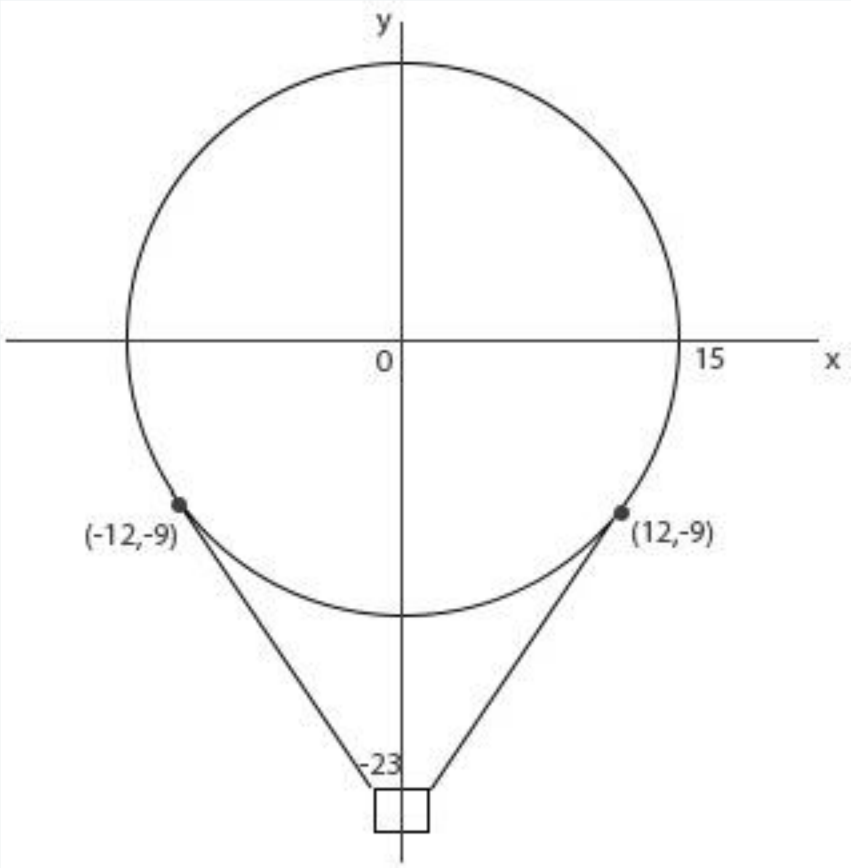
We can find the linear equation of that cable by using line equation:

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y + 9 = -\frac{4}{3}(x + 12)$$

$$y = -\frac{4}{3}x - 25$$

Question 1



Question 1

The value of x at the left side of the box can be known by substituting the above equation with y = -23

$$x_{left} = -1.5$$

Repeating whole procedure above to the right cable, we will get

$$x_{right} = 1.5$$

Therefore, the width of the gondola is 3

Question 1

c. From physics, we know that

$$r = v.t$$

$$r = \sqrt{64(h-y)}\,.\,t$$

t is time taken for the water to fall and reach the base, which is equal to

$$t = \sqrt{\frac{2y}{g}}$$

therefore, we get the required equation:

$$r = \sqrt{64(h-y).\frac{2y}{g}}$$

$$r = \sqrt{\frac{128}{g}(hy-y^2)}$$

$$r = \sqrt{\frac{128}{g}}\,.\left(hy-y^2\right)^{\frac{1}{2}}$$

Question 1

To get the maximum value of r, $\frac{dr}{dt} = 0$

$$\frac{dr}{dt} = \sqrt{\frac{128}{g}} \cdot \frac{1}{2}(hy - y^2)^{-\frac{1}{2}} \cdot (h - 2y) = 0$$

$$\frac{dr}{dt} = h - 2y = 0$$

$$y = \frac{h}{2}$$

Question 2

a. $\frac{d(mv)}{dt} = F + (v+u)\frac{dm}{dt}$

Substituting the given equation with the given variable, we have

$$\frac{d(mv)}{dt} = -mg - (v-c)b$$

$\frac{d(mv)}{dt}$ can be expanded into

$$\frac{dm}{dt}v + \frac{dv}{dt}m$$

Then the equation becomes

$$-bv + \frac{dv}{dt}m = -mg - vb + cb$$

$$\frac{dv}{dt} = \frac{-mg + cb}{m} = -g + \frac{cb}{m}$$

because $\frac{dm}{dt} = -b$, $m = -bt + C$

at t = 0, $m = m_0$ so $C = m_0$

Therefore, we have

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Question 2

$$\frac{dv}{dt} = -g + \frac{cb}{-bt + m_0}$$

$$dv = \left(-g + \frac{cb}{-bt + m_0}\right) dt$$

by integrating both sides, we have

$$v = -gt - u \ln\left(\frac{m_0 - bt}{m_0}\right) + C_1$$

Initially, v = 0, so C_1 = 0

because $v = \frac{dy}{dt}$,

$$y = \int_0^t -gt - u \ln\left(\frac{m_0 - bt}{m_0}\right) dt$$

$$y = -\frac{1}{2}gt^2 + c\left[t + \left(\frac{m_0 - bt}{b}\right) \ln\left(\frac{m_0 - bt}{m_0}\right)\right]$$

Question 2

b. $\frac{dy}{dt} = k\frac{A}{V}(c - y)$

$$\frac{dy}{y-c} = -\frac{kA}{v}dt$$

$$\int \frac{dy}{y-c} = -\frac{kA}{v}\int dt$$

$$\ln|y-c| = -\frac{kA}{v}t + C_1$$

$$y - c = C_2 e^{-\frac{kA}{v}t}$$

By applying the initial condition $y(0) = y_0$,

$$y_0 = C_2 + c$$

Therefore,

$$y = (y_0 - c)e^{-\frac{kA}{v}t} + c$$

Steady-state solution:

$$y(\infty) = c$$

Question 3

a. Let x_1, y_1, z_1 be on the plane $Ax + By + Cz = D_1$,
let x, y, z be on the plane $Ax + By + Cz = D_2$
let $\overrightarrow{QP_1} = (x - x_1)\boldsymbol{i} + (y - y_1)\boldsymbol{j} + (z - z_1)\boldsymbol{k}$
and let $\boldsymbol{n} = \frac{A\boldsymbol{i}+B\boldsymbol{j}+C\boldsymbol{k}}{\sqrt{A^2+B^2+C^2}}$
The distance is

$$\left|proj_{\boldsymbol{n}}\, \overrightarrow{QP_1}\right| = \left|\left((x - x_1)\boldsymbol{i} + (y - y_1)\boldsymbol{j} + (z - z_1)\boldsymbol{k}\right) . \left(\frac{A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}}{\sqrt{A^2 + B^2 + C^2}}\right)\right|$$

$$= \frac{\left|Ax_1 + By_1 + Cz_1 - (Ax + By + Cz)\right|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D_1 - D_2|}{|A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}|}$$

Question 3

b. Substituting the above equation with given value

$$d = \frac{|12 - 6|}{\sqrt{4 + 9 + 1}} = \frac{6}{\sqrt{14}}$$

c. $\frac{|2(3)+(-1)(2)+2(-1)+4|}{\sqrt{14}} = \frac{|2(3)+(-1)(2)+2(-1)-D|}{\sqrt{14}}$

D = 8 or -4

Desired plane:

2x – y + 2x = 8

d. Choose the point (2, 0, 1) on the plane. Then

$$\frac{|3 - D|}{\sqrt{6}} = 5$$

$$D = 3 \pm 5\sqrt{6}$$

Desired plane:

$$x - 2y + z = 3 + 5\sqrt{6}$$

and

$$x - 2y + z = 3 - 5\sqrt{6}$$

Question 4

a.

$$z^3 = \frac{-4+4i}{1-\sqrt{3}i} \cdot \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$$

$$z^3 = \frac{-\left(4+4\sqrt{3}\right)+\left(4-4\sqrt{3}\right)i}{4} = -1-\sqrt{3}+\left(1-\sqrt{3}\right)i$$

$$r = \sqrt{\left(-1-\sqrt{3}\right)^2+\left(1-\sqrt{3}\right)^2}$$

$$r = 2\sqrt{2}$$

$$\theta = \tan^{-1}\frac{1-\sqrt{3}}{-1-\sqrt{3}}$$

$$\theta = -\frac{11}{12}\pi$$

$$z^3 = 2\sqrt{2}e^{-\frac{11}{12}\pi i}$$

$$z = \sqrt[3]{2\sqrt{2}e^{-\frac{11}{36}\pi i}}$$

$$z = 1.414e^{-\frac{11}{36}\pi i}$$

$$z = 1.414(\cos\left(-\frac{11}{36}\pi\right)+i\sin\left(-\frac{11}{36}\pi\right))$$

Question 4

b. (i) for α:

$$r = \sqrt{1+3} = 2$$
$$\theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$
$$\alpha = 2(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right))$$

for β:

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$
$$\theta = \tan^{-1}\left(\frac{-\frac{1}{2}}{\frac{1}{2}}\right) = -\frac{\pi}{4}$$
$$\beta = \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

(ii) $\alpha^3\beta^4 = \left(2e^{\frac{\pi}{3}i}\right)^3\left(\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}i}\right)^4$

$$= 8e^{\pi i}.\frac{1}{4}e^{-\pi i}$$
$$= 2$$

(iii) $P(z) = (z-\alpha)(z-\alpha^*)(z-\beta)(z-\beta^*)$

$$P(z) = (z^2 - 2z + 4)(z^2 - z + \frac{1}{2})$$