BBMerlion

Vector

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Definition & representation

- Vector are quantities which possess both **direction and magnitude.** Example of vectors are acceleration, velocity, and displacement.

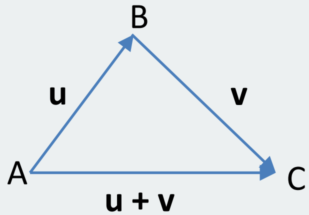
- Vectors can be represented such as follows :
- Directed segment($\overrightarrow{AB}$) : represents vector from point A to point B

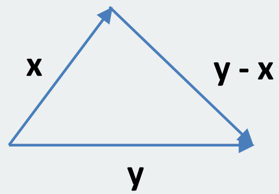
$$A \longrightarrow B$$

- $\left\|\overrightarrow{AB}\right\|$: represents the length/magnitude of $\overrightarrow{AB}$

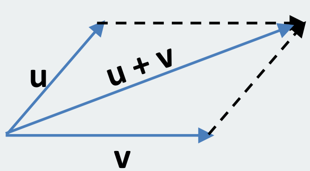
Addition and substraction

- Let **u** = $\overrightarrow{AB}$ and **v** = $\overrightarrow{BC}$, then **u + v =** $\overrightarrow{AB}$ + $\overrightarrow{BC}$ $=$ $\overrightarrow{AC}$.
- Let **x** = $\overrightarrow{AB}$ and **y** = $\overrightarrow{AC}$, then for **y – x** = $\overrightarrow{AC}$ - $\overrightarrow{AB}$ = $\overrightarrow{AC}$ + $\overrightarrow{BA}$ = $\overrightarrow{BC}$.





- Parallelogram Law of Vector Addition



Scalar multiplication

- For any real number **μ** and a given vector **u,** we called **μu** as vector multiplication.
  - If k = 0 then **μu** is a zero vector.
  - If k > 0 then **μu** is in the initial direction of **u** but with magnitude **μ-**times of |u|
  - If k < 0 then **μu** is in the opposite direction of **u** but with magnitude **μ-**times of |u|

Vectors in coordinate system

- We may represent/position a vector in 2-D plane ($\mathbb{R}^2$) or 3-D space ($\mathbb{R}^3$), with its initial point at the origin O and terminal point at *x, y,* and z (in case of $\mathbb{R}^2$ only at x and y).
- **v =** x$\hat{\boldsymbol{\imath}}$ **+** y$\hat{\boldsymbol{\jmath}}$ **+** z$\widehat{\boldsymbol{k}}$
- Where $\hat{\boldsymbol{\imath}}$**,** $\hat{\boldsymbol{\jmath}}$**,** and $\widehat{\boldsymbol{k}}$ are called **unit vectors** and x, y, and z are called **components** of **v.** We may also represent vectors in **column vector form**:
- **v =** $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}$

Length or norm of A VECTOR (modulus)

- Assume there are 2 points in space A = $(x_1, y_1,\ z_1)$ and B = $(x_2, y_2,\ z_2)$. Thus the norm of vector $\overrightarrow{AB}$ = $(x_2 - x_1, y_2 -\ y_1,\ z_2 -\ z_1)$ is given by:
  - $||AB|| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- A vector of length 1 is called **unit vector.** Generally we may determine the unit vector as the following:
  - $\hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{||\boldsymbol{u}||}$

Position vector of a point dividing a given line in a given ratio

- Assuming a line AB, and the location of point A and B w.r.t O is **a** and **b** (**a and b** is in coordinate form). If there is a point C which divide AB; AC:BC = $\lambda : \mu$, then the location of point C is given by;
    - $C = \frac{\lambda . \boldsymbol{b} + \mu . \boldsymbol{a}}{\lambda + \mu}$

Dot product (scalar product)

- The dot product of 2 non-zero vectors **u,** and **v** is defined by:
  - **u.v** = $||\boldsymbol{u}|| \, ||\boldsymbol{v}|| \cos\theta$
- The dot product of 2 non-zero vectors is scalar (i.e it only has magnitude and no direction). The dot product is also **commutative** i.e it does not matter if we calculate **u.v** or **v.u,** either way will have the same result (unlike the cross product which we will discuss later in this chapter).

Dot product in coordinate form

- Let two vectors in space **u** = (x_1, y_1, z_1) and **v** = (x_2, y_2, z_2). The dot product of **u.v** is given by:
  - $\boldsymbol{u}.\boldsymbol{v} = x_1.x_2 + y_1.y_2 + z_1.z_2$

Properties of dot product

- 1. The dot product of two vectors is a scalar.
- 2. u.v = v.u (Commutative)
- 3. For each vector u, u.u $=||u||^2$.
- 4. Suppose that u and v are non-zero vectors. Then u.v = 0 $\Leftrightarrow$ u $\perp$ **v**
- 5. u.(v + w) = u.v + u.w:

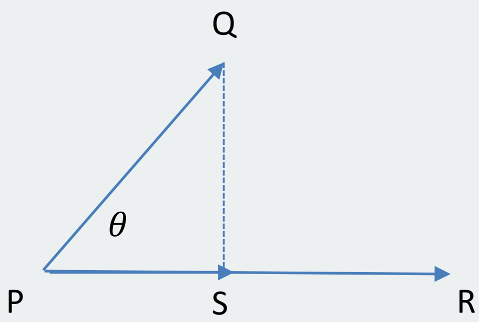
Application of dot product: ANGLE BETWEEN VECTORS

- By manipulating the given formula of dot product we are able to determine the angle between vectors if both vectors are known.

- $\cos\theta = \dfrac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$

Application of dot product: projection

- **u** = $\overrightarrow{PQ}$ and **v** = $\overrightarrow{PR}$, then $\overrightarrow{PS}$ is the projection of **u onto v.**



$$\|\overrightarrow{PS}\| = \|\boldsymbol{u}\| \cos\theta \text{ and } \cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

Thus,

$$\|\overrightarrow{PS}\| = \frac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

- Notice that, we just calculated the length of $\overrightarrow{PS}$ and not the vector solution. To determine the vector (sometimes denoted as; $proj_{\boldsymbol{v}}\boldsymbol{u}$) we only need to multiply the norm of $\overrightarrow{PS}$ with unit vector $\widehat{\boldsymbol{v}}$.

- $\overrightarrow{PS} = \frac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{v}\|}\, \widehat{\boldsymbol{v}}$

Cross product (vector product)

- The cross product (or vector product) of two vectors **u** and **v** in $\mathbb{R}^3$ is defined as:
  - $\boldsymbol{u} \times \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta\, \hat{\boldsymbol{n}}$
- Where θ is the angle between **u** and **v**, and $\hat{\boldsymbol{n}}$ is the vector perpendicular to both **u** and **v** governed by the right hand rule.

Cross product in coordinate form

- Let two vectors in space **u** = (x_1, y_1, z_1) and **v** = (x_2, y_2, z_2). The cross product of u and v can be determined by using the determinant formula:

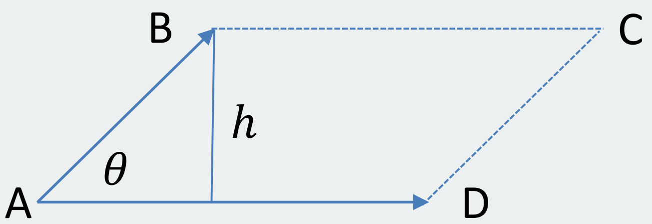
- $\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

Properties of cross product

- 1. The cross product of two vectors u and v is a vector perpendicular to both u and v (if it is non-zero).
- 2. If u and v are parallel, then = 0 and thus u x v = 0:
- 3. u x v = -(v x u) (anti-commutative).
- 4. ux(v + w) = u x v + u x w (distributive w.r.t. addition).
- 5. (**k.**u)x(**l.**v) = (k.l)(u x v): (where k and l are constants).

Application of cross product: area

- By using the cross product formula, we are able to calculate the area of parallelogram ABCD and triangle ABD.



$$
\begin{aligned}
&\text{area of parallelogram } ABCD \\
&= base.height \\
&= \|AD\|\|AB\| \sin\theta \\
&= \|\overrightarrow{AD} \times \overrightarrow{AB}\|
\end{aligned}
$$

$$
\begin{aligned}
&\text{area of triangle } ABD \\
&= \frac{1}{2}\|\overrightarrow{AD} \times \overrightarrow{AB}\|
\end{aligned}
$$

lines

- We have learnt that in Cartesian plane a line may be determined by its gradient and a point on the line. In space, line is uniquely determined by its **direction vector** and **point on the line. The vector equation** of line *l is*
- $l : r(t) =$ **coordinaate of point on line + k . direction vector**

- To get a better understanding, let's take a look at the example on the next slide.

lines

- Example:
- Given 3 points in space A (1,0,1), B (0,1,1), and C (1,1,0). Determine the vector equation of a line in space which is parallel to $\overrightarrow{AB}$ and passes through C.
- $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
- Thus the vector equation of the line is;
- $l = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k.\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

lines

- Besides the vector equation, we may represent the line in its Cartesian equation. Assuming l = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, point on the line = $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ and the direction vector **v** = $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Thus,
- $x = x_0 + kv_1, y = y_0 + kv_2, and\ z = z_0 + kv_3$. These are called **parametric equation** of l. The **Cartesian equation** is defined below:
  - $\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$

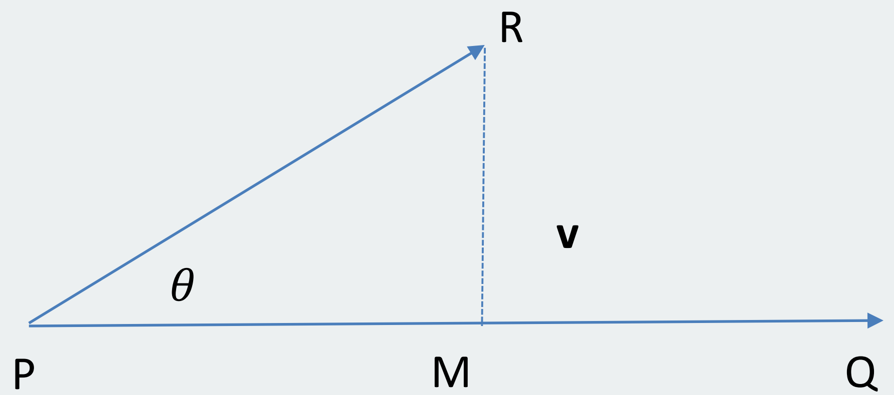
LINES: ANGLE BETWEEN TWO LINES

- We have learnt that by using dot product formula we are able to determine angle between 2 vectors. Same thing goes with angle between two lines. Assuming two lines with direction vector $\mathbf{v_1}$ and $\mathbf{v_2}$. By performing dot product between those two direction vectors we will get the angle between the lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$.

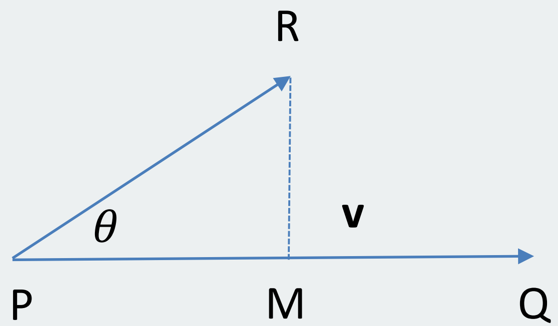
- $\theta = \cos^{-1}\left|\frac{v_1.v_2}{\|v_1\|\|v_2\|}\right|$

Lines: distance from a point to a line

- Assuming a line which passes through a point P and with direction vector **v.** If there is any point R such that it is not located on the line, we are able to calculate the distance RM of that point to the line l.



Lines: distance from a point to a line



- Distance of point R to line l (RM)
- $= \|PR\| \sin\theta$
- $= \|PR\| \dfrac{\|PR \times \mathbf{v}\|}{\|PR\|\|\mathbf{v}\|}$
- $= \dfrac{\|PR \times \mathbf{v}\|}{\|\mathbf{v}\|}$

plane

- In space a plane is represented by a point on the plane and a normal vector **n** which is perpendicular to the plane.
- If there is a point $P_0 = (x_0, y_0, z_0)$, a normal non-zero vector **n** = (a, b, c) and any arbitrary point P = (x, y, z) which we assume located on the plane. We can represent the plane in its **vector equation** =
  - $\boldsymbol{n}.\overrightarrow{P_0 P} = 0$
  - $\begin{pmatrix} a \\ b \\ c \end{pmatrix} . \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$

plane

- If we expand the vector equation of the plane, we will get:
  - $a.(x - x_0) + \mathrm{b}.(y - y_0) + \mathrm{c}.(z - z_0) = 0$
- From this we may get the **scalar equation of the plane** which is :
  - $ax + by + cz = a.x_0 + b.y_0 + c.z_0$
    - And can be simplified as
      - $ax + by + cz = d$

Plane: angle between two planes

- In plane, the angle between to planes is angle between the respective normal vectors $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ which is quite similiar to the case of angle between two lines;

- $\theta = \cos^{-1}\left|\frac{\boldsymbol{n}_1.\boldsymbol{n}_2}{\|\boldsymbol{n_1}\|\|\boldsymbol{n}_2\|}\right|$

Plane: distance from a point to a plane

- Assuming a plane with a point P_0 on the plane and a normal **n,** if there is any arbitrary point Q which is not on the plane, we are able to determine the distance from the point to the plane namely h
- distance from point Q to the plane (h)
- $= \left\|\overrightarrow{P_0Q}\right\| \cos\theta \,, where\ \theta\ is\ angle\ between\ \overrightarrow{P_0Q}\ and\ \boldsymbol{n}$
- $= \left\|\overrightarrow{P_0Q}\right\| \left(\frac{\overrightarrow{P_0Q}.\boldsymbol{n}}{\left\|\overrightarrow{P_0Q}\right\|\|\boldsymbol{n}\|}\right)$
- $= \frac{\overrightarrow{P_0Q}.\boldsymbol{n}}{\|\boldsymbol{n}\|}$

Pairs of lines

- The location of two lines in space may be such that either it's 1) **parallel**, 2) **not parallel and intersect**, 3) not parallel and not intersect (called **skew**).
- The lines are called **coplanar** if they are located in the same plane.
- The lines are parallel if one direction vector is plus/minus integral multiplication of another direction vector; $\boldsymbol{v_1} = \mu \boldsymbol{v_2}$.
- The lines will intersect if we can find a point where these lines meet. Assume two line vector equation $\boldsymbol{r_1} = a_0 + k_1 \boldsymbol{v_1}$, and $\boldsymbol{r_2} = b_0 + k_1 \boldsymbol{v_2}$. If we are able to find k_1 and k_2 (these constants does not necessary to be equal with each other) such that $a_0 + k_1 \boldsymbol{v_1} = b_0 + k_1 \boldsymbol{v_2}$. Then, the lines are intersect.

A line and a plane

- The angle between a line with direction vector **v** and a plane with normal **n** is given by;
  - $\sin\theta = \frac{\boldsymbol{n}.\boldsymbol{v}}{||\boldsymbol{n}||\,||\boldsymbol{v}||}$